R. S. HUTCHEON

GRAPHICAL AND MECHANICAL COMPUTATION

PART II. EXPERIMENTAL DATA

BY

JOSEPH LIPKA, PH.D.

LATE ASSISTANT PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY

NEW YORK
JOHN WILEY & SONS, INC.
LONDON: CHAPMAN & HALL, LIMITED

IN THE REPRINTING OF THIS BOOK, THE RECOMMENDATIONS OF THE WAR PRODUCTION BOARD HAVE BEEN OBSERVED FOR THE CONSERVATION OF PAPER AND OTHER IMPORTANT WAR MATERIALS. THE CONTENT REMAINS COMPLETE AND UNABRIDGED.

Printed in U. S. A.

8-43

PREFACE

This book embodies a course given by the writer for a number of years in the Mathematical Laboratory of the Massachusetts Institute of Technology. It is designed as an aid in the solution of a large number of problems which the engineer, as well as the student of engineering, meets in his work.

In the opening chapter, the construction of scales naturally leads to a discussion of the principles upon which the construction of various slide rules is based. The second chapter develops the principles of a network of scales, showing their application to the use of various kinds of coördinate paper and to the charting of equations in three variables.

Engineers have recognized for a long time the value of graphical charts in lessening the labor of computation. Among the charts devised none are so rapidly constructed nor so easily read as the charts of the alignment or nomographic type — a type which has been most fully developed by Professor M. d'Ocagne of Paris. Chapters III, IV, and V aim to give a systematic development of the construction of alignment charts; the methods are fully illustrated by charts for a large number of well-known engineering formulas. It is the writer's hope that the simple mathematical treatment employed in these chapters will serve to make the engineering profession more widely acquainted with this time and labor saving device.

Many formulas in the engineering sciences are empirical, and the value of many scientific and technical investigations is enhanced by the discovery of the laws connecting the results. Chapter VI is concerned with the fitting of equations to empirical data. Chapter VII considers the case where the data are periodic, as in alternating currents and voltages, sound waves, etc., and gives numerical, graphical, and mechanical methods for determining the constants in the equation.

When empirical formulas cannot be fitted to the experimental data, these data may still be efficiently handled for purposes of further computation, — interpolation, differentiation, and integration, — by the numerical, graphical, and mechanical methods developed in the last two chapters.

Numerous illustrative examples are worked throughout the text, and a large number of exercises for the student is given at the end of each chapter. The additional charts at the back of the book will serve

as an aid in the construction of alignment charts. Bibliographical references will be found in the footnotes.

The writer wishes to express his indebtedness for valuable data to the members of the engineering departments of the Massachusetts Institute of Technology, and to various mathematical and engineering publications. He owes the idea of a Mathematical Laboratory to Professor E. T. Whittaker of the University of Edinburgh. He is especially indebted to Capt. H. M. Brayton, U. S. A., a former student, for his valuable suggestions and for his untiring efforts in designing a large number of the alignment charts. Above all he is most grateful to his wife for her assistance in the revision of the manuscript and the reading of the proof, and for her constant encouragement which has greatly lightened the labor of writing the book.

JOSEPH LIPKA.

CAMBRIDGE, MASS.,
Oct. 13, 1918.

CONTENTS.

CHAPTER I.

SCALES AND THE SLIDE RULE.

CHAPTER II.

NETWORK OF SCALES. CHARTS FOR EQUATIONS IN TWO AND THREE VARIABLES.

CHAPTER III.

NOMOGRAPHIC OR ALIGNMENT CHARTS.

CHAPTER IV.

NOMOGRAPHIC OR ALIGNMENT CHARTS (Continued).

CHAPTER V.

NOMOGRAPHIC OR ALIGNMENT CHARTS (Continued).

CHAPTER VI.

EMPIRICAL FORMULAS — NON-PERIODIC CURVES.

CHAPTER VII.

EMPIRICAL FORMULAS — PERIODIC CURVES.

CHAPTER VIII.

INTERPOLATION.

CHAPTER IX.

APPROXIMATE INTEGRATION AND DIFFERENTIATION.

CHAPTER VI.

EMPIRICAL FORMULAS — NON-PERIODIC CURVES.

68. Experimental data. — In scientific or technical investigations we are often concerned with the observation or measurement of two quantities, such as the distance and the time for a freely falling body, the volume of carbon dioxide dissolving in water and the temperature of the water, the load and the elongation of a certain wire, the voltage and the current of a magnetite arc, etc. The results of a series of measurements of the same two quantities under similar conditions are usually presented in the form of a table. Thus the following table gives the results of observations on the pressure p of saturated steam in pounds per sq. in. and the volume v in cu. ft. per pound:

$p =$	10	20	30	40	50	60
$v =$	37.80	19.72	13.48	10.29	8.34	6.62

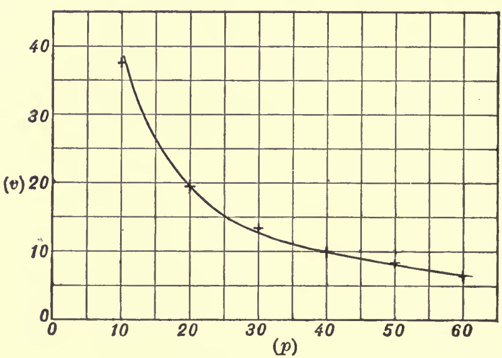


FIG. 68.

We represent these results graphically by plotting on coördinate paper the points whose coördinates are the corresponding values of the measured quantities and by drawing a smooth curve through or very near these points. Fig. 68 gives a graphical representation of the above table, where the values of p are laid off as abscissas and the values of v as ordinates and a smooth curve is drawn so as to pass through or very near the plotted points.

The fact that a smooth curve can be drawn so as to pass very near the plotted points leads us to suspect that some relation may exist between the measured quantities, which may be represented mathematically by the equation of the curve. Since the original measurements, the plotting of the points, and the drawing of the curve all involve approximations, the equation will represent the true relation between the quantities only approximately. Such an equation or formula is known as an *empirical formula*, to distinguish it from the equation or formula which expresses a physical, chemical, or biological law. A large number of the formulas in the engineering sciences are empirical formulas. Such empirical formulas may then be used for the purpose of interpolation, *i.e.*, for computing the value of one of the quantities when the value of the other is given within the range of values used in determining the formula.

It is at once evident that any number of curves can be drawn so as to pass very near the plotted points, and therefore that any number of equations might approximate the data equally well. The nature of the experiment may give us a hint as to the form of the equation which will best represent the data. Otherwise the problem is more indeterminate. If the points appear to lie on or near a straight line, we may assume an equation of the first degree, $y = a + bx$, in the variables. But if the points deviate systematically from a straight line, the choice of an equation is more difficult. Often the form of the curve will suggest the type of equation, parabolic, exponential, trigonometric, etc., but in all cases, we should choose an equation of as simple a form as possible. Before proceeding any further with this choice we may test the correctness of the form of the equation by "rectifying" the curve, *i.e.*, by writing the assumed equation in the form

$$(1)\ f(y) = a + bF(x) \qquad \text{or} \qquad (2)\ y' = a + bx',$$

where $y' = f(y)$ and $x' = F(x)$, and plotting the points with x' and y' as coördinates; if the points of this plot appear to lie on or very near a straight line, then this line can be represented by equation (2) and hence the original curve by equation (1). We shall use the method of rectification quite freely in the work which follows.

Having chosen a simple form for the approximate equation we now proceed to determine the approximate values of the constants or coefficients appearing in the equation. The method of approximation employed in determining these constants depends upon the desired degree of accuracy. We may employ one of three methods: the *method of selected points*, the *method of averages*, or the *method of Least Squares*. Of these, the first is the simplest and the approximation is close enough for a large number of problems arising in technical work; the second requires a little more computation but usually gives closer approximations;

while the third gives the best approximate values of the constants but the work of determining these values is quite laborious. All three methods will be illustrated in some of the problems which follow.

After the constants have been determined the formula should be tested by performing several additional experiments where the variables lie within the range of the previous data, and comparing these results with those given by the empirical formula.

We shall now work two illustrative examples to indicate the general method of procedure.

(I) THE STRAIGHT LINE.

69. The straight line, $y = bx$. — The following table gives the results of a series of experiments on the determination of the elongation E in inches of annealed high carbon steel wire of diameter 0.0693 in. and gage length 30 in. due to the load W in pounds.

W	E	EW	W^2	E_c^{I}	E_c^{II}	E_c^{III}	Δ^{I}	Δ^{II}	Δ^{III}
0	0	0	0	0	0	0	0	0	0
50	0.0130	0.650	2,500	0.0130	0.0131	0.0131	0	− 1	− 1
100	0.0251	2.510	10,000	0.0260	0.0261	0.0262	− 9	−10	−11
150	0.0387	5.805	22,500	0.0390	0.0392	0.0393	− 3	− 5	− 6
200	0.0520	10.400	40,000	0.0520	0.0522	0.0524	0	+ 2	− 4
225	0.0589	13.253	50,625	0.0585	0.0587	0.0589	+ 4	+ 2	0
250	0.0659	16.475	62,500	0.0650	0.0653	0.0655	+ 9	+ 6	+ 4
260	0.0689	17.914	67,600	0.0676	0.0679	0.0681	+13	+10	+ 8
Σ 1235	0.3425	67.007	255,725				38	36	34
						$\Sigma \div 8 =$	4.8	4.5	4.3
						$\Sigma\Delta^2 =$	356	270	254

The plot. — The data are plotted on a sheet of coördinate paper about 10 inches square and ruled in twentieths of an inch or in millimeters. If we wish to express the elongation as a function of the load, we plot the load on the horizontal axis or as abscissas, if the load as a function of the elongation we plot the latter as abscissas. In Fig. 69 we have plotted the values of W as abscissas and the values of E as ordinates. The scales with which these values are plotted are generally chosen so that the length of the axis represents the total range of the corresponding variable, and so that the line or curve is about equally inclined to the two axes. There is no advantage in choosing the scale units on the two axes equal. Care should be taken not to choose the units either too small or too large; for in the former case the precision of the data will not be utilized, and in the latter case the deviations from a representative line

or curve are likely to be magnified. The drawing of a good plot is evidently a matter of judgment. It is best to mark the plotted points as the intersection of two short straight lines, one horizontal and one vertical.

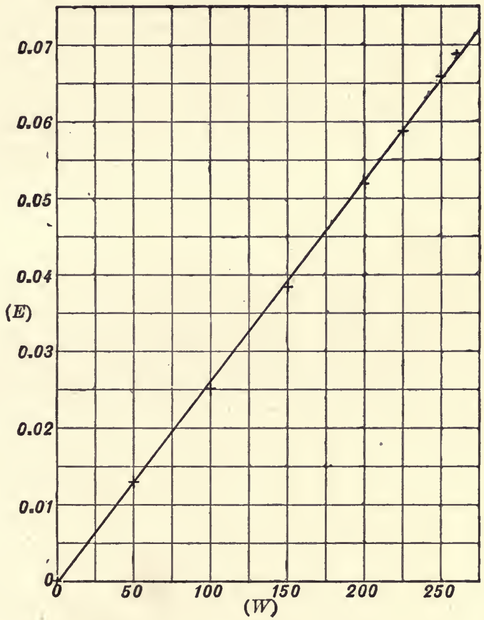


FIG. 69.

The representative curve and its equation. — We now draw a smooth curve passing very near to the points of the plot, so that the deviations of the points from the curve are very small, some positive and some negative. In Fig. 69, the points seem to fall approximately on a straight line. This should be tested by moving a stretched thread or by sliding a sheet of celluloid with a fine line scratched on its under side among the points and noting that the points do not deviate systematically from this thread or line. Having decided that a straight line will approximate the plot, we assume that an equation of the first degree, $E = a + bW$, will approximately represent the relation between the measured quantities. In this example we may evidently assume that $E = bW$ since a zero load gives a zero elongation.

The determination of the constant. — We shall now determine the constant b in the equation $E = bW$. This may be done in several ways. The three methods which are generally employed are as follows:

I. *Method of selected points.* — Place the sheet of celluloid on the coördinate paper so that the scratched line passes through the point $W = 0$, $E = 0$, and then rotate the sheet until a good average position among the plotted points is obtained, *i.e.*, until the largest possible number of points lie either on the line or alternately on opposite sides of the line, in such a manner that the points below the line deviate from it by approximately the same amount as the points above it. Then note the values of W and E corresponding to one other point on this line, preferably near the farther end of the line. Thus we read $W = 250$, $E = 0.0650$. Substituting these values in the equation $E = bW$, we have $0.0650 = 250\,b$, and hence $b = 0.000260$, and finally $E = 0.000260\,W$. Since the choice of the "best" line is a matter of judgment, its position, and hence the value of the constant, will vary with different workers and often with the same worker at different times.

II. *Method of averages.* — The vertical distances of the plotted points from the representative line are called the *residuals;* these are the differences between the observed values of E and the values of E calculated from the formula, or $E - E_c$, where $E_c = bW$; some of these residuals are positive and others are negative. If we assume that the "best" line is that which makes the algebraic sum of the residuals equal to zero, we have

$$\Sigma\,(E - bW) = 0 \quad \text{or} \quad \Sigma E - b\Sigma W = 0,$$

hence

$$b = \frac{\Sigma E}{\Sigma W} = \frac{0.3225}{1235} = 0.000261,$$

and we may call this an average value of b. By this method it is no longer necessary to shift the line among the points so as to get an average position.

III. *Method of Least Squares.* — In the theory of Least Squares * it is shown that the best line or the best value of the constant is that which makes the sum of the squares of the differences of the observed and calculated values a minimum, *i.e.*,

$$\Sigma\,(E - bW)^2 = \text{minimum}.$$

Hence the derivative of this expression with respect to b must equal zero, or

$$\frac{d}{db}\,\Sigma\,(E - bW)^2 = 0, \quad \text{or} \quad \Sigma\,W(E - bW) = 0,$$

or

$$\Sigma WE - b\Sigma W^2 = 0, \quad \text{and} \quad b = \frac{\Sigma EW}{\Sigma W^2}.$$

* See Bartlett's "The Method of Least Squares," or any other book on this theory.

We form two columns, one giving the values of EW and the other the values of W^2, and adding these columns, we find

$$b = 67.007/255{,}725 = 0.000262.$$

We may now compare the results obtained by each of the three methods. For this purpose we complete the table by computing the values of E from the formulas

$$\text{I. } E = 0.000260\, W; \quad \text{II. } E = 0.000261\, W; \quad \text{III. } E = 0.000262\, W.$$

These are marked E_c^{I}, E_c^{II}, E_c^{III}, in the table. To discover how closely the computed values agree with the observed values we form the residuals

$$\Delta^{\text{I}} = E - E_c^{\text{I}}, \qquad \Delta^{\text{II}} = E - E_c^{\text{II}}, \qquad \Delta^{\text{III}} = E - E_c^{\text{III}}.$$

Disregarding the signs of these residuals, we add them and divide by their number, 8, and find the average residual to be 0.00048, 0.00045, 0.00043, respectively. We also find the sum of the squares of the residuals to be 356, 270, 254, respectively. We may therefore draw the following conclusions: all three methods give good results; the method of Least Squares gives the best value of the constant but requires the most calculation; the method of averages gives, in general, the next best value of the constant and requires but little calculation; the graphical method of selected points requires the least calculation but depends upon the accuracy of the plot and the fitting of the representative line.

70. The straight line, $y = a + bx$. — For measuring the temperature coefficient of a copper rod of diameter 0.3667 in. and length 30.55 in., the following measurements were made. Here, C is the temperature Centigrade and r is the resistance of the rod in microhms.

C	r	C^2	rC	r_c^{I}	r_c^{II}	r_c^{III}	Δ^{I}	Δ^{II}	Δ^{III}
19.1	76.30	364.81	1,457.33	76.19	76.19	76.26	+0.11	+0.11	+0.04
25.0	77.80	625.00	1,945.00	77.91	77.92	77.96	−0.11	−0.12	−0.16
30.1	79.75	906.01	2,400.48	79.39	79.41	79.43	+0.36	+0.34	+0.32
36.0	80.80	1296.00	2,908.80	81.11	81.14	81.13	−0.31	−0.34	−0.33
40.0	82.35	1600.00	3,294.00	82.27	82.31	82.28	+0.08	+0.04	+0.07
45.1	83.90	2034.01	3,783.89	83.75	83.80	83.76	+0.15	+0.10	+0.14
50.0	85.10	2500.00	4,255.00	85.18	85.24	85.16	−0.08	−0.14	−0.06
Σ 245.3	566.00	9325.83	20,044.50				1.20	1.19	1.12
						Σ ÷ 7 =	0.171	0.170	0.160
						Σ Δ² =	2852	2869	2646

The plot (Fig. 70) appears to approximate a straight line, so that we shall assume the relation $r = a + bC$. We shall determine the constants, a and b, by the three methods.

I. *Method of selected points.* — Use a sheet of celluloid to determine the approximate position of the best straight line, and note two points

on this line; thus, $C = 20$, $r = 76.45$, and $C = 48$, $r = 84.60$. Substituting these values in the equation $r = a + bC$, we get

$$76.45 = a + 20\,b \quad \text{and} \quad 84.60 = a + 48\,b,$$

from which we determine

$$a = 70.63 \quad \text{and} \quad b = 0.291,$$

so that our relation becomes

$$r = 70.63 + 0.291\,C.$$

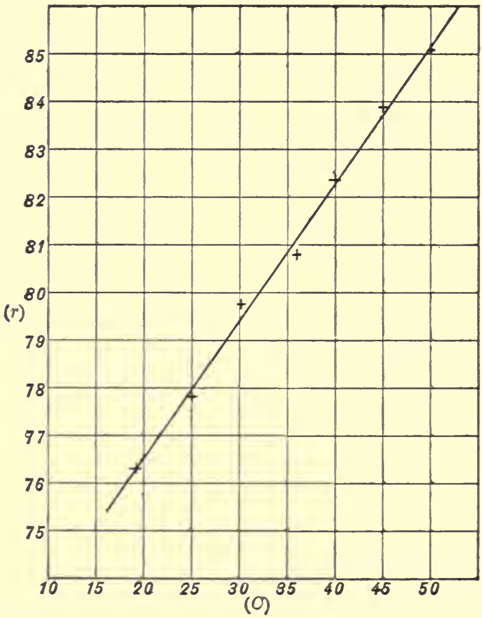


FIG. 70.

II. *Method of averages.* — Since we have to determine two constants, we divide the data into two equal or nearly equal groups, and place the sum of the residuals in each group equal to zero, *i.e.*,

$$\Sigma\,(r - a - bC) = 0 \quad \text{or} \quad \Sigma r = na + b\Sigma C,$$

where n is the number of observations in the group. Thus, dividing the above data into two groups, the first containing four and the second three sets of data, and adding, we get

$$314.65 = 4\,a + 110.2\,b \quad \text{and} \quad 251.35 = 3\,a + 135.1\,b,$$

from which we determine

$$a = 70.59 \quad \text{and} \quad b = 0.293,$$

so that our relation becomes

$$r = 70.59 + 0.293\, C.$$

III. *Method of Least Squares.* — The best values of the constants are those for which the sum of the squares of the residuals is a minimum, *i.e.*, $\Sigma\,(r - a - bC)^2 =$ minimum; hence the partial derivatives of this expression with respect to a and b must be zero; thus,

$$\frac{\partial}{\partial a}\Sigma\,(r - a - bC)^2 = 0, \quad \frac{\partial}{\partial b}\Sigma\,(r - a - bC)^2 = 0,$$

or $\quad \Sigma\,[2\,(r - a - bC)\,(-1)] = 0, \qquad \Sigma\,[2\,(r - a - bC)\,(-C)] = 0,$

or

$$\Sigma r = an + b\Sigma C,$$
$$\Sigma rC = a\Sigma C + b\Sigma C^2,$$

where n is the number of observations. We solve these last two equations for a and b. (Note that these equations may be formed as follows: substitute the observed values of r and C in the assumed relation $r = a + bC$; add the n equations thus formed to get the first of the above equations; multiply each of the n equations by the corresponding value of C and add the resulting n equations to get the second of the above equations.)

We now compute the values of rC, C^2, ΣC, ΣrC, and ΣC^2, and substitute these in the equations for determining a and b. We thus get

$$566.00 = 7\,a + 245.3\,b,$$
$$20{,}044.50 = 245.3\,a + 9325.83\,b,$$

from which we determine

$$a = 70.76 \quad \text{and} \quad b = 0.288,$$

so that our relation becomes

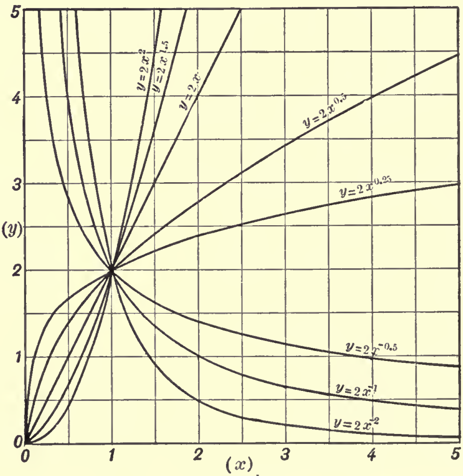
$$r = 70.76 + 0.288\, C.$$

Comparison of results. — We note that the various results agree very well with the original data and with each other. We compute the residuals and find that the average residual is smallest by the third method and is approximately the same by the first two methods. The computation necessary in applying the method of Least Squares is very tedious. The method of selected points requires the fitting of the best straight line, and this becomes quite difficult when the number of plotted points is large. We shall therefore use the method of averages in most of the illustrative examples which follow.

(II) FORMULAS INVOLVING TWO CONSTANTS.

71. Simple parabolic and hyperbolic curves, $y = ax^b$. — As stated in Art. 68, when the plotted points deviate systematically from a straight line, a smooth curve is drawn so as to pass very near the points; the shape of the curve or a knowledge of the nature of the experiment may give us a hint as to the form of the equation which will best represent the data.

Simple curves which approximate a large number of empirical data are the parabolic and hyperbolic curves. The equation of such a curve is $y = ax^b$, parabolic for b positive and hyperbolic for b negative. In Fig. 71a, we have drawn some of these curves for $a = 2$ and $b = -2$, -1, -0.5, 0.25, 0.5, 1.5, 2. Note that the parabolic curves all pass



$y = ax^b$

FIG. 71a.

through the points (0, 0) and (1, a) and that as one of the variables increases the other increases also. The hyperbolic curves all pass through the point (1, a) and have the coördinate axes as asymptotes, and as one of the variables increases the other decreases.

There is a very simple method of verifying whether a set of data can be approximated by an equation of the form $y = ax^b$. Taking logarithms of both members of this equation, we get $\log y = \log a + b \log x$, and if $x' = \log x$, $y' = \log y$, this becomes $y' = \log a + bx'$, an equation of the first degree in x' and y'; therefore the plot of (x', y') or of $(\log x, \log y)$ must approximate a straight line. Hence,

If a set of data can be approximately represented by an equation of the form $y = ax^b$, then the plot of (log x, log y) approximates a straight line.

Instead of plotting (log x, log y) on ordinary coördinate paper, we may plot (x, y) directly on logarithmic coördinate paper (see Art. 13). We determine the constants a and b from the equation of the straight line by one of the methods described in Art. 70.

Example. The following table gives the number of grams S of anhydrous ammonium chloride which dissolved in 100 grams of water makes a saturated solution of $\theta°$ absolute temperature.

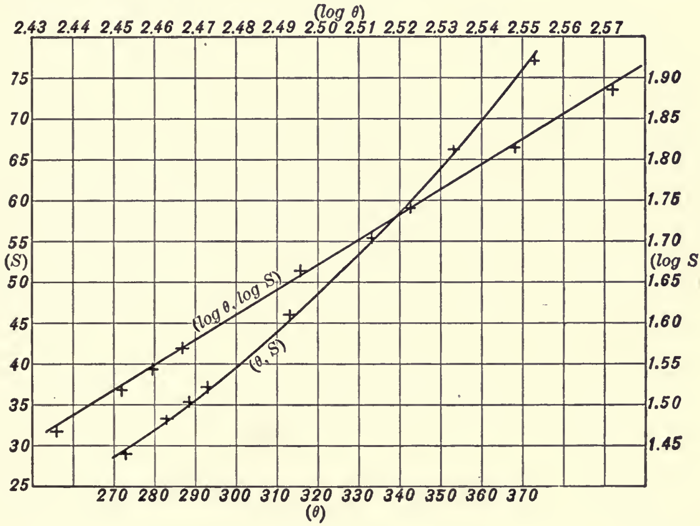


FIG. 71*b*.

θ	S	$\log\theta$	$\log S$	S_e^{I}	S_e^{II}	Δ^{I}	Δ^{II}
273	29.4	2.4362	1.4684	29.7	29.7	−0.3	−0.3
283	33.3	2.4518	1.5224	33.2	33.2	+0.1	+0.1
288	35.2	2.4594	1.5465	35.0	35.1	+0.2	+0.1
293	37.2	2.4669	1.5705	37.0	37.0	+0.2	+0.2
313	45.8	2.4955	1.6609	45.3	45.3	+0.5	+0.5
333	55.2	2.5224	1.7419	54.9	54.9	+0.3	+0.3
353	65.6	2.5478	1.8169	65.7	65.8	−0.1	−0.2
373	77.3	2.5717	1.8882	77.9	78.0	−0.6	−0.7
					$\Sigma \div 8 =$	0.29	0.30

The points (θ, S) are plotted in Fig. 71*b*. The curve appears to be parabolic, *i.e.*, of the general form illustrated in Fig. 71*a*. We therefore plot $(\log \theta, \log S)$ and note that this approximates a straight line, so that we may assume

$$S = a\theta^b \quad \text{or} \quad \log S = \log a + b \log \theta.$$

We shall first determine the constants by the method of selected points. We note two points on the line whose coördinates are

$$\log \theta = 2.445, \ \log S = 1.50 \quad \text{and} \quad \log \theta = 2.555, \ \log S = 1.84,$$

hence we have

$$1.50 = \log a + 2.445\, b,$$
$$1.84 = \log a + 2.555\, b.$$

$$\therefore \quad b = 3.09, \quad \log a = -6.0550 = 3.9450 - 10, \quad a = 0.000{,}000{,}881.$$
$$\therefore \quad \log S = -6.0550 + 3.09 \log \theta, \quad \text{or} \quad S = 0.000{,}000{,}881\, \theta^{3.09}.$$

We shall now determine the constants by the method of averages. We divided the data into two groups of four sets, and adding, we have

$$6.1078 = 4 \log a + \ \ 9.8143\, b,$$
$$7.1079 = 4 \log a + 10.1374\, b.$$

$$\therefore\ b = 3.09, \ \log a = -6.0546 = 3.9454 - 10, \ a = 0.000000882.$$
$$\therefore\ \log S = -6.0546 + 3.09 \log \theta \quad \text{or} \quad S = 0.000000882\, \theta^{3.09}.$$

We complete the table by computing S, the residuals, and the average residual. The agreement between the observed and computed values of S is quite close.

Example. The following table gives the pressure p in pounds per sq. in. of saturated steam corresponding to the volume v in cu. ft. per pound. (From Perry's Elementary Practical Mathematics.)

v	p	$\log v$	$\log p$	p_c	Δ
53.92	6.86	1.7318	0.8363	6.85	+0.01
26.36	14.70	1.4210	1.1673	14.69	+0.01
14.00	28.83	1.1461	1.4599	28.85	−0.02
6.992	60.40	0.8446	1.7810	60.49	−0.09
4.280	101.9	0.6314	2.0082	102.1	−0.2
2.748	163.3	0.4390	2.2130	163.7	−0.4
1.853	250.3	0.2679	2.3984	249.2	+1.1

The points (v, p) are plotted in Fig. 71*c*. The curve appears to be hyperbolic on comparison with Fig. 71*a*. Hence we plot $(\log v, \log p)$ and note that this approximates a straight line, so that we may assume

$$p = av^b, \quad \text{or} \quad \log p = \log a + b \log v.$$

We shall use the method of averages to determine the constants a and b.

Dividing the data into two groups, the first four and the last three sets, and adding, we have

$$5.2445 = 4 \log a + 5.1435\, b,$$
$$6.6196 = 3 \log a + 1.3383\, b.$$

$\therefore \quad b = -1.0662, \quad \log a = 2.6822, \quad a = 481.1.$

$\therefore \quad \log p = 2.6822 - 1.0662 \log v, \quad$ or $\quad pv^{1.0662} = 481.1.$

We now compute p and Δ and note the close agreement between the observed and calculated values.

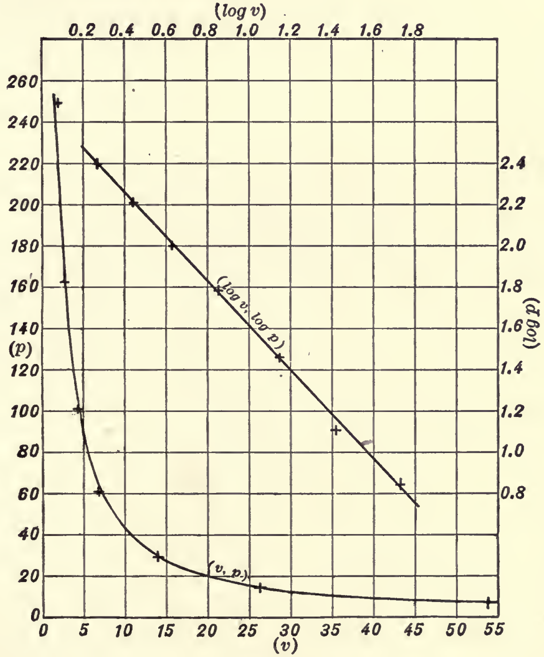
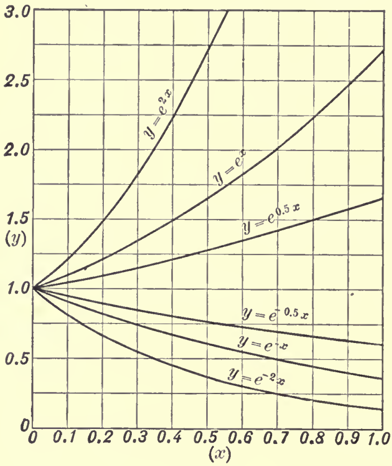


FIG. 71c.

72. Simple exponential curves, $y = ae^{bx}$. — Other simple curves that approximate a large number of experimental results are the exponential or logarithmic curves. The equation of such a curve may be written in the form $y = ae^{bx}$, where e is the base of natural logarithms; the form $y = ab^{x}$ is sometimes used. In Fig. 72a, we have drawn some of these curves for $a = 1$ and $b = -2, -1, -0.5, 0.5, 1, 2$. Note that these curves all pass through the point $(0, a)$ and have the x-axis for asymptote.

There is a very simple method of verifying whether a set of data can be approximated by an equation of the form $y = ae^{bx}$. Taking logarithms of both members of this equation we get $\log y = \log a + (b \log e) x$, and if $y' = \log y$, this equation becomes $y' = \log a + (b \log e) x$, an equation of the first degree in x and y'; therefore the plot of (x, y') or of $(x, \log y)$ must approximate a straight line. Hence,



$y = ae^{bx}$

FIG. 72*a*.

If a set of data can be approximately represented by an equation of the form $y = ae^{bx}$, then the plot of $(x, \log y)$ approximates a straight line.

Instead of plotting $(x, \log y)$ on ordinary coördinate paper, we may plot (x, y) directly on semilogarithmic coördinate paper (see Art. 14). The constants a and b are determined from the equation of the straight line by one of the methods described in Art. 70.

Example. Chemical experiments by Harcourt and Esson gave the results of the following table, where A is the amount of a substance remaining in a reacting system after an interval of time t.

t	A	$\log t$	$\log A$	A_c	Δ
2	94.8	0.3010	1.9768	94.9	−0.1
5	87.9	0.6990	1.9440	87.7	+0.2
8	81.3	0.9031	1.9101	81.0	+0.3
11	74.9	1.0414	1.8745	74.8	+0.1
14	68.7	1.1461	1.8370	69.1	−0.4
17	64.0	1.2304	1.8062	63.8	+0.2
27	49.3	1.4314	1.6928	49.0	+0.3
31	44.0	1.4914	1.6435	44.1	−0.1
35	39.1	1.5441	1.5922	39.6	−0.5
44	31.6	1.6435	1.4997	31.2	+0.4

$\Sigma\Delta \div 10 = 0.26$

The points (t, A) are plotted in Fig. 72*b*. This curve appears to be exponential, so that we plot $(t, \log A)$ and $(\log t, A)$; it is seen that the plot of $(t, \log A)$ approximates a straight line. We may therefore assume an equation of the form

$$A = ae^{bt} \quad \text{or} \quad \log A = \log a + (b \log e) t.$$

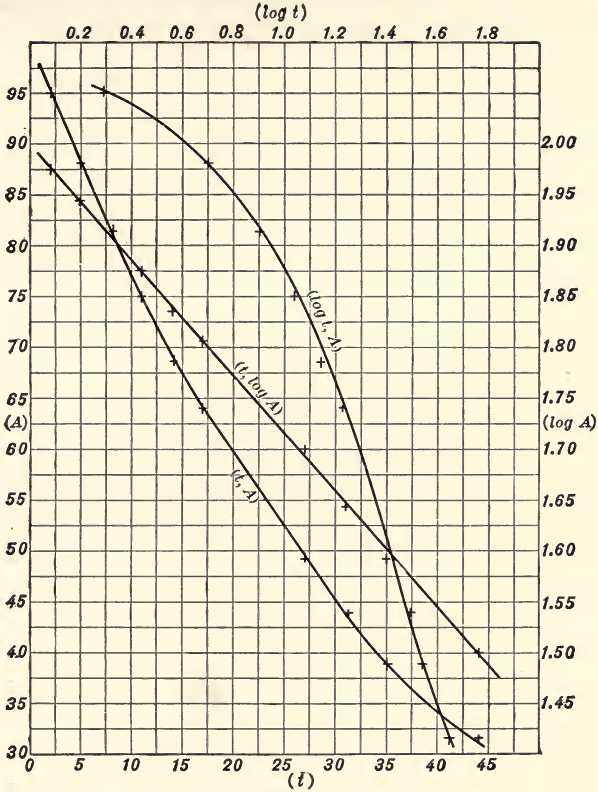


FIG. 72b.

We shall use the method of averages to determine the constants. Dividing the data into 2 groups and adding, we get

$$9.5424 = 5 \log a + 40 \, (b \log e),$$
$$8.2344 = 5 \log a + 154 \, (b \log e).$$
$$\therefore \quad b \log e = -0.0115, \quad \log a = 2.0005.$$
$$\therefore \quad b = -0.0265, \qquad a = 100.1, \text{ since } \log e = 0.4343.$$
$$\therefore \quad \log A = 2.0005 - 0.0115\, t, \quad \text{or} \quad A = 100.1\, e^{-0.0265\, t}.$$

We now compute the values of A and the residuals, and note the close agreement between the observed and the calculated values of A.

Example. The following table gives the results of measuring the electrical conductivity C of glass at temperature $\theta°$ Fahrenheit.

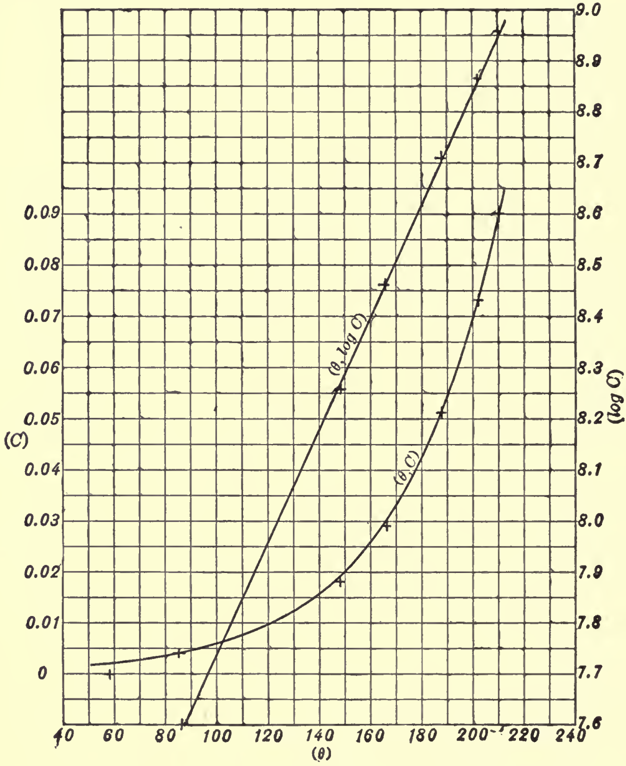


FIG. 72c.

θ	C	$\log\theta$	$\log C$	C_e	Δ
58	0	1.7634	$-\infty$	0.0019	
86	0.004	1.9345	7.6021 − 10	0.0039	+0.0001
148	0.018	2.1703	8.2553 − 10	0.0185	−0.0005
166	0.029	2.2201	8.4624 − 10	0.0292	−0.0002
188	0.051	2.2742	8.7076 − 10	0.0510	0
202	0.073	2.3054	8.8633 − 10	0.0728	+0.0002
210	0.090	2.3222	8.9542 − 10	0.0891	−0.0010

In Fig. 72*c*, the points (θ, C) and $(\theta, \log C)$ are plotted; the latter plot approximates a straight line. We may therefore assume the equation

$$C = ae^{b\theta}, \quad \text{or} \quad \log C = \log a + (b \log e)\,\theta.$$

We use the method of averages to determine the constants. Omitting the first set and dividing the remaining data into two groups of three sets, we get

$$24.3198 - 30 = 3 \log a + 400\,(b \log e),$$
$$26.5251 - 30 = 3 \log a + 600\,(b \log e).$$
$$\therefore \quad b \log e = 0.0110, \quad \log a = 6.6399 - 10.$$
$$\therefore \quad b = 0.0253, \qquad a = 0.000436.$$
$$\therefore \quad \log C = 6.6399 - 10 + 0.0110\,\theta, \quad \text{or} \quad C = 0.00436\, e^{0.0253\,\theta}.$$

We now compute the values of C and the residuals and note the remarkably close agreement between the observed and computed values of C.

73. Parabolic or hyperbolic curve, $y = a + bx^n$ (where n is known).— In using this equation, it is assumed that from theoretical considerations we suspect the value of n. It is evident that

If a set of data can be approximately represented by an equation of the form $y = a + bx^n$, where n is known, then the plot of (x^n, y) approximates a straight line.

Example. A small condensing triple expansion steam engine tested under seven steady loads, each lasting three hours, gave the following results; I is the indicated horse-power, w is the number of pounds of steam used per hour per indicated horse-power. (From Perry's Elementary Practical Mathematics.)

I	w	wI	w_c	Δ
36.8	12.5	460.0	12.6	−0.1
31.5	12.9	406.4	12.8	+0.1
26.3	13.1	344.5	13.0	+0.1
21.0	13.3	279.3	13.4	−0.1
15.8	14.1	222.8	14.0	+0.1
12.6	14.5	182.7	14.6	−0.1
8.4	16.3	136.9	16.1	+0.2

$\Sigma\Delta \div 7 = 0.11$

Fig. 73*a* gives the plot of (I, w). This is not a straight line. But if we plot (I, wI), *i.e.*, the total weight of steam used per hour instead of the weight per indicated horse-power, we find that this plot approximates a straight line. Hence, we may assume the linear relation $wI = a + bI$. This relation may also be written $w = b + a/I$, so that the plot of $(1/I, w)$ also approximates a straight line. We use the method of averages to

determine the constants. Dividing the data into two groups, the first three and last four sets, and adding, we have

$$1210.9 = 3\,a + 94.6\,b,$$
$$821.7 = 4\,a + 57.8\,b.$$
$$\therefore \quad b = 11.6, \quad a = 37.8.$$
$$\therefore \quad wI = 37.8 + 11.6\,I, \quad \text{or} \quad w = 11.6 + \frac{37.8}{I}.$$

We now compute the values of w and the residuals.

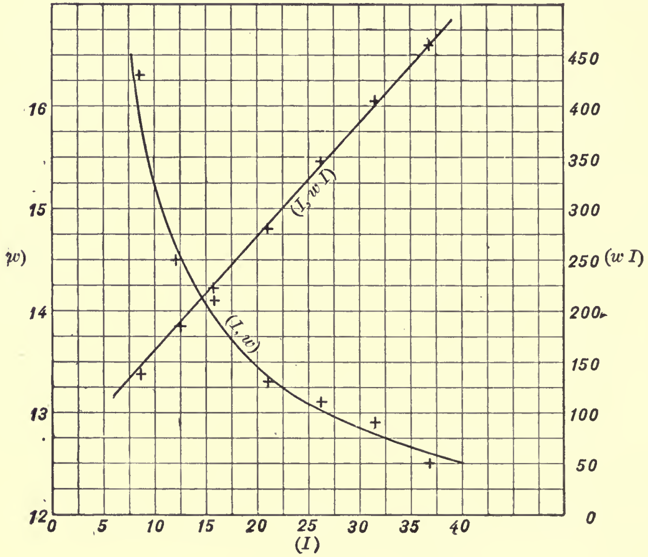


FIG. 73a.

Example. For a parachute or flat plate falling in air we have the following observations; v is the velocity in ft. per sec. and p is the pressure in pounds per sq. in.

v	p	v^2	p_c	Δ
7.87	0.2	61.94	0.187	−0.013
11.50	0.4	132.25	0.401	+0.001
16.40	0.8	268.96	0.815	−0.015
22.60	1.6	510.76	1.548	+0.052
32.80	3.2	1075.84	3.260	−0.060

$\Sigma\Delta \div 5 = 0.028$

In Fig. 73*b*, we have plotted (v, p). It is surmised that for low velocities, the pressure and the square of the velocity are linearly related, *i.e.*, $p = a + bv^2$. We verify this by plotting (v^2, p) and noting that this approximates a straight line. We use the method of averages to determine the constants. Dividing the data into two groups, the first three and the last two sets, and adding, we have

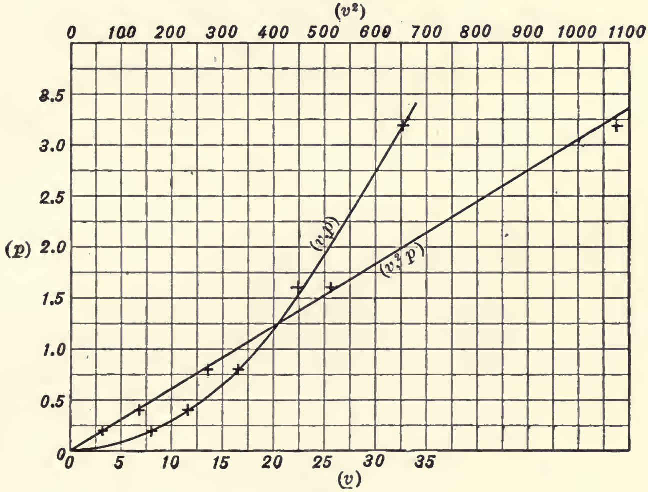


FIG. 73*b*.

$$1.4 = 3\,a + \;\;463.15\,b,$$
$$4.8 = 2\,a + 1586.60\,b.$$

$$\therefore \quad b = 0.00303 \quad \text{and} \quad a = -0.00111.$$
$$\therefore \quad p = -0.00111 + 0.00303\,v^2.$$

We may with good approximation take $a = 0$, so that $p = 0.00303\,v^2$, *i.e.*, the pressure varies directly as the square of the velocity.

74. Hyperbolic curve, $y = \dfrac{x}{a + bx}$, **or** $\dfrac{x}{y} = a + bx$. —This equation represents the ordinary hyperbola with asymptotes $x = -a/b$ and $y = 1/b$, as illustrated in Fig. 74*a* for values of $a = 0.2$, $b = 0.2$; $a = 0.1$, $b = 0.2$; $a = -0.1$, $b = 0.2$; $a = -0.2$, $b = 0.2$. Quite a large number of experimental results may be represented by an equation of this type.

The equation may also be written in the form $\frac{1}{y} = b + \frac{a}{x}$, so that the plots $\left(x, \frac{x}{y}\right)$ and $\left(\frac{1}{x}, \frac{1}{y}\right)$ approximate straight lines. Hence,

If a set of data can be approximately represented by an equation of the form $y = \frac{x}{a + bx}$, *or* $\frac{x}{y} = a + bx$ *then the plot of* $\left(x, \frac{x}{y}\right)$ *or of* $\left(\frac{1}{x}, \frac{1}{y}\right)$ *approximates a straight line.*

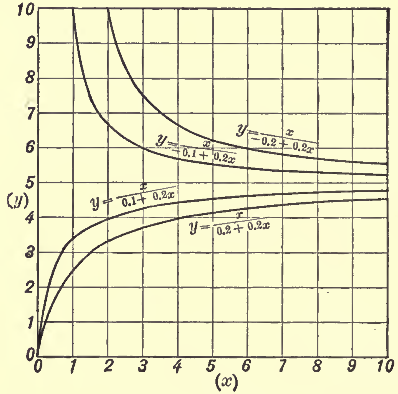


FIG. 74*a*. $y = \frac{x}{a + bx}$

Example. From a magnetization or normal induction curve for iron we find the following data; H is the number of Gilberts per cm., a measure of the field intensity, and B is the number of kilolines per sq. cm., a measure of the flux density.

H	B	H/B	B_c	Δ
2.5	3.5	0.714	7.97	
3.0	5.0	0.600	8.78	
3.1	7.5	0.413	8.91	
3.8	10.0	0.380	9.8	+0.2
7.0	12.5	0.560	12.4	+0.1
9.5	13.5	0.703	13.6	−0.1
11.3	14.0	0.808	14.0	0
17.5	15.0	1.17	15.1	−0.1
31.5	16.0	1.97	16.2	−0.2
45.0	16.5	2.72	16.7	−0.2
64.0	17.0	3.76	17.0	0
95.0	17.5	5.43	17.3	+0.2

$\Sigma\Delta \div 9 = 0.12$

In Fig. 74*b*, (H, B) is plotted. The curve appears to be of the type illustrated in Fig. 74*a*. Furthermore, an important quantity in the

theory of magnetization is the reluctivity H/B, and if we plot $(H, H/B)$, we note that this plot approximates a straight line for values of $H > 3.1$. (We may similarly introduce the permeability, B/H, and note that the plot of $(B/H, B)$ approximates a straight line.) Hence, we assume a relation of the form $\frac{H}{B} = a + bH$. Using the method of averages,

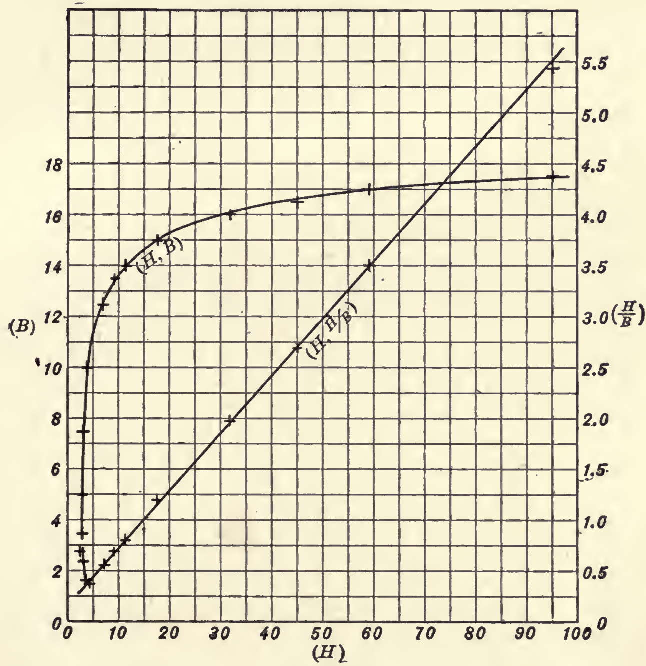


FIG. 74*b*.

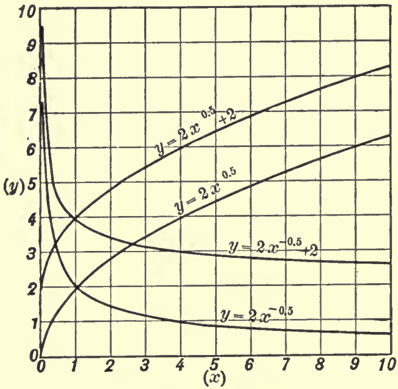
omitting the first three values of H, and dividing the remaining data into two groups containing five and four sets respectively, we get the equations

$$3.621 = 5a + 49.1\,b,$$
$$13.88 = 4a + 235.5\,b.$$
$$\therefore \quad b = 0.0560, \quad a = 0.174.$$
$$\therefore \quad \frac{H}{B} = 0.174 + 0.0560\,H \quad \text{or} \quad B = \frac{H}{0.174 + 0.0560\,H}.$$

We now compute B and the residuals and note the close agreement between the observed and computed values.

(III) FORMULAS INVOLVING THREE CONSTANTS.

75. The parabolic or hyperbolic curve, $y = ax^b + c$. — It is often impossible to fit a simple equation involving only *two* constants to a set of data. In such cases we may modify our simple equations by the addition of a term involving a third constant. Thus the equation $y = ax^b$ may be modified into $y = ax^b + c$. If b is positive, the latter equation represents a parabolic curve with intercept c on OY; if b is negative, the equation represents a hyperbolic curve with asymptote $y = c$. In Fig. 75*a*, we have sketched the curves $y = 2x^{0.5}$, $y = 2x^{0.5} + 2$, $y = 2x^{-0.5}$, $y = 2x^{-0.5} + 2$ to illustrate the relation of the simple types to the modified types.



$y = ax^b + c$

FIG. 75*a*.

In Art. 71 it was shown that if we suspect a relation of the form $y = ax^b$, we can verify this by observing whether the plot of ($\log x$, $\log y$) approximates a straight line. Now the form $y = ax^b + c$ may be written $\log (y - c) = \log a + b \log x$, so that the plot of ($\log x$, $\log (y - c)$) would approximate a straight line. To make this test we shall evidently first have to determine a value of c. We might attempt to read the value of c from the original plot of (x, y). In the parabolic case we should have to read the intercept of the curve on OY, but this may necessitate the extension of the curve beyond the points plotted from the given data, a procedure which is not safe in most cases. In the hyperbolic case, we should have to estimate the position of the asymptote, but this is generally a difficult matter.

The following procedure will lead to the determination of an approximate value of c for the equation $y = ax^b + c$. Choose two points (x_1, y_1) and (x_2, y_2) on the curve sketched to represent the data. Choose a third point (x_3, y_3) on this curve such that $x_3 = \sqrt{x_1 x_2}$, and measure the value of y_3. Then, since the three points are on the curve, their coördinates must satisfy the equation of the curve, so that

$$y_1 = ax_1^b + c, \quad y_2 = ax_2^b + c, \quad y_3 = ax_3^b + c.$$

Now, since $x_3 = \sqrt{x_1 x_2}$,

therefore $x_3^b = \sqrt{x_1^b x_2^b}$, and $ax_3^b = \sqrt{ax_1^b \cdot ax_2^b}$,

or $y_3 - c = \sqrt{(y_1 - c)(y_2 - c)}$,

and therefore $c = \dfrac{y_1 y_2 - y_3^2}{y_1 + y_2 - 2\,y_3}$.

It is evident that the determination of c is partly graphical, for it depends upon the reading of the coördinates of three points on the curve sketched to represent the data. The curve should be drawn as a smooth line lying evenly among the points, *i.e.*, so that the largest number of the plotted points lie on the curve or are distributed alternately on opposite sides and very near it.

Having determined a value for c, we plot $(\log x, \log (y - c))$. If this plot approximates a straight line, the constants a and b in the equation $\log (y - c) = \log a + b \log x$ may then be determined in the ordinary way.

Example. In a magnetite arc, at constant arc length, the voltage V consumed by the arc is observed for values of the current i. (From Steinmetz, Engineering Mathematics.)

i	V	$V - 30.4$	$\log (V - 30.4)$	$\log i$	V_c	Δ
0.5	160	129.6	2.1126	9.6990 − 10	158.8	+1.2
1	120	89.6	1.9523	0.0000 − 10	120.8	−0.8
2	94	63.6	1.8035	0.3010 − 10	94.0	0
4	75	44.6	1.6493	0.6021 − 10	75.1	−0.1
8	62	31.6	1.4997	0.9031 − 10	61.9	+0.1
12	56	25.6	1.4082	1.0792 − 10	56.0	0

We plot (i, V) and note that the curve appears hyperbolic with an asymptote $V = c$, and hence we assume an equation of the form $V = ai^b + c$. To verify this we must first determine a value for c. Choose two points on the experimental curve; in Fig. 75*b*, we read $i_1 = 0.5$, $V_1 = 160$ and $i_2 = 12$, $V_2 = 56$. Choose a third point such that $i_3 = \sqrt{i_1 i_2} = \sqrt{6} = 2.45$, and measure $V_3 = 88$. Then

$$c = \frac{V_1 V_2 - V_3^2}{V_1 + V_2 - 2\,V_3} = \frac{(160)(56) - (88)^2}{160 + 56 - 2\,(88)} = \frac{1216}{40} = 30.4.$$

Now compute the values of $V - 30.4$ and $\log (V - 30.4)$ and plot $(\log i, \log (V - 30.4))$. This last plot approximates a straight line so that the choice of the equation $V = ai^b + c$ is verified.

To determine the constants in the equation

$$\log (V - 30.4) = \log a + b \log i,$$

we use the method of averages, dividing the data into two groups of three sets each, and find

$$5.8684 = 3 \log a,$$
$$4.5572 = 3 \log a + 2.5844\, b.$$
$$\therefore \quad b = -0.507, \quad \log a = 1.9561, \quad a = 90.4.$$
$$\therefore \quad \log (V - 30.4) = 1.9561 - 0.507 \log i, \text{ or } V = 30.4 + 90.4\, i^{-0.507}.$$

Finally, we compute the values of V and the residuals.

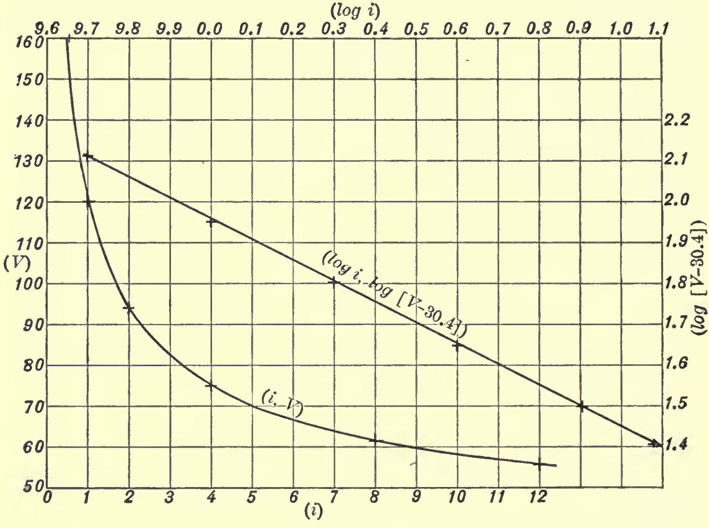


FIG. 75*b*.

76. The exponential curve, $y = ae^{bx} + c$. — The simple exponential equation $y = ae^{bx}$ may have to be modified into $y = ae^{bx} + c$ in order to fit a given set of data. In the latter curve, the asymptote is $y = c$. In Fig. 76*a*, we have sketched the curves $y = 2\, e^{0.1x}$, $y = 2\, e^{0.1x} + 1$, $y = 2\, e^{-0.1x}$, $y = 2\, e^{-0.1x} + 1$.

In Art. 72 it was shown that if we suspect a relation of the form $y = ae^{bx}$, we can verify this by observing whether the plot of $(x, \log y)$ approximates a straight line. Now $y = ae^{bx} + c$ may be written $\log (y - c) = \log a + (b \log e)\, x$, so that the plot of $(x, \log (y - c))$ would approximate a straight line. Evidently we shall first have to determine a value for c. We proceed to do this in a manner similar to that employed in Art. 75. Choose two points (x_1, y_1) and (x_2, y_2) on

the curve sketched to represent the data, and then a third point (x_3, y_3) on this curve such that $x_3 = \frac{1}{2}(x_1 + x_2)$ and measure the value of y_3. Since the three points are on the curve.

$$y_1 = ae^{bx_1} + c, \qquad y_2 = ae^{bx} + c, \qquad y_3 = ae^{bx_3} + c,$$

or $\log \dfrac{y_1 - c}{a} = (b \log e)\, x_1,\ \log \dfrac{y_2 - c}{a} = (b \log e)\, x_2,\ \log \dfrac{y_3 - c}{a} = (b \log e)\, x_3.$

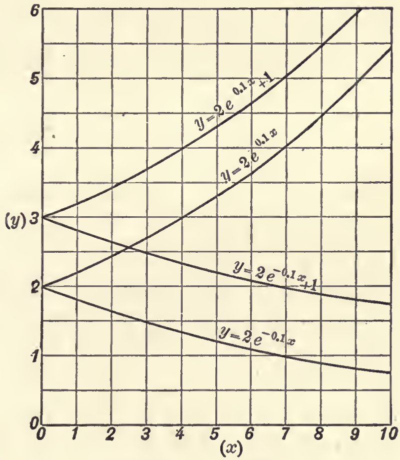


FIG. 76a. $y = ae^{bx} + c$

Now, since $\qquad x_3 = \frac{1}{2}(x_1 + x_2),$

therefore $\qquad (b \log e)\, x_3 = \frac{1}{2}[(b \log e)\, x_1 + (b \log e)\, x_2],$

and $\quad \log \dfrac{y_3 - c}{a} = \dfrac{1}{2}\left[\log \dfrac{y_1 - c}{a} + \log \dfrac{y_2 - c}{a}\right] = \log \sqrt{\dfrac{y_1 - c}{a} \cdot \dfrac{y_2 - c}{a}}.$

Hence $\quad y_3 - c = \sqrt{(y_1 - c)(y_2 - c)}$, and $\quad c = \dfrac{y_1 y_2 - y_3^2}{y_1 + y_2 - 2y_3}.$

If the data are given so that the values of x are equidistant, *i.e.*, so that they form an arithmetic progression, we may verify the choice of the equation $y = ae^{bx} + c$ and determine the constants a, b, and c in the following manner. Let the constant difference in the values of x equal h. If we replace x by $x + h$, we get $y' = ae^{b(x+h)} + c$, and therefore, for the difference in the values of y,

$$\Delta y = y' - y = ae^{b(x+h)} - ae^{bx} = ae^{bx}(e^{bh} - 1),$$

and $$\log \Delta y = \log a\,(e^{bh} - 1) + (b \log e)\, x.$$

This last equation is of the first degree in x and $\log \Delta y$ so that the plot of $(x, \log \Delta y)$ is a straight line. To apply this to our data, we form a column of successive differences, Δy, of the values of y, and a column of the logarithms of these differences, $\log \Delta y$, and plot $(x, \log \Delta y)$; if the equation $y = ae^{bx} + c$ approximates the data, then this last plot will approximate a straight line. We may then determine $b \log e$ and $\log a\,(e^{bh} - 1)$ and hence a and b in the ordinary way, and finally find an average value of c from $\Sigma y = a\Sigma e^{bx} + nc$, where n is the number of data.

Example. In studying the skin effect in a No. 0000 solid copper conductor of diameter 1.168 cm., Kennelly, Laws, and Pierce found the following experimental results; F is the frequency in cycles per second, L is the total abhenrys observed.

F	L	$L - 51{,}860$	$\log (L - 51{,}860)$	L_c	Δ
60	53,912	2052	3.3122	53,952	−40
306	53,767	1907	3.2804	53,668	+99
888	53,143	1283	3.1082	53,140	+ 3
1600	52,669	809	2.9079	52,699	−30
2040	52,499	639	2.8055	52,506	− 7
3065	52,215	355	2.5502	52,212	+ 3
3950	52,082	222	2.3464	52,068	+14
5000	51,965	105	2.0212	51,972	− 7

In Fig. 76*b*, the points (F, L) are plotted; the curve appears to be exponential with an asymptote $L = c$. We shall try to fit the equation $L = ae^{bF} + c$. First determine an approximate value for c by choosing two points on the experimental curve, $F_1 = 875$, $L_1 = 53{,}140$, and $F_2 = 5000$, $L_2 = 51{,}980$, and a third point $F_3 = \frac{1}{2}\,(F_1 + F_2) = 2938$, $L_3 = 52{,}250$. Then $c = \dfrac{L_1L_2 - L_3^2}{L_1 + L_2 - 2\,L_3} = 51{,}860$. Now compute $(L - 51{,}860)$ and $\log (L - 51{,}860)$, and plot $(F, \log (L - 51{,}860))$; this plot approximates a straight line, thus verifying the choice of equation. We determine the constants in the equation $\log (L - 51{,}860) = \log a + (b \log e)\,F$ by the method of averages. Dividing the data into two groups of four sets each and adding, we have

$$12.6087 = 4 \log a + 2854\, b \log e,$$
$$9.7233 = 4 \log a + 14{,}055\, b \log e.$$

$$\therefore \quad b \log e = -0.0002576, \quad \log a = 3.3360,$$

and
$$b = -0.0005931, \quad a = 2168.$$

$$\therefore \quad \log (L - 51{,}860) = 3.3360 - 0.0002576\, F,$$

or
$$L = 51{,}860 + 2168\, e^{-0.0005931\, F}.$$

We now compute L and the residuals, and note the close agreement between the observed and computed values except for the first two values of F. If we omit these two values in computing a and b, these constants have slightly different values, but the agreement between the observed and computed values of L is about the same.

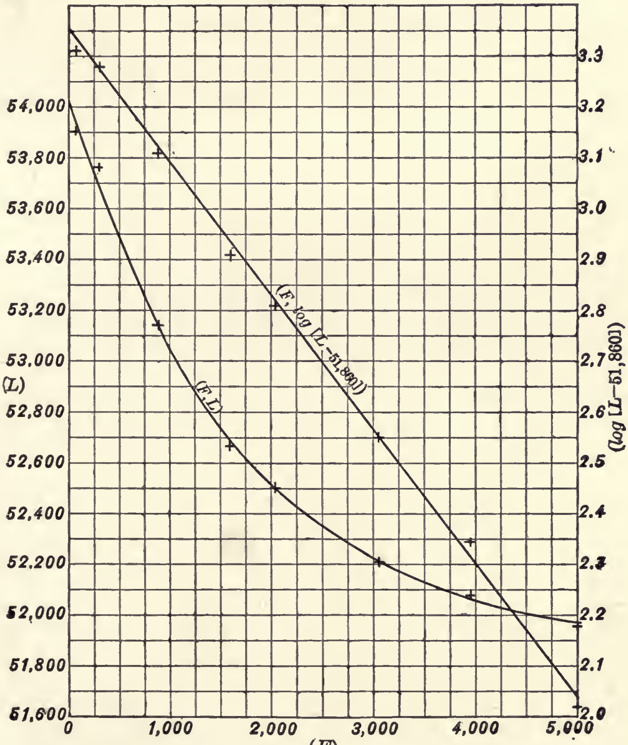


FIG. 76b.

77. The parabola, $y = a + bx + cx^2$. — The equation of the straight line $y = a + bx$ may be modified by the addition of a term of the second degree to the form $y = a + bx + cx^2$. This is the equation of the ordinary parabola. We may verify whether this equation fits a set of experimental data by one of the following methods.

(1) Choose any point (x_k, y_k) on the experimental curve; then $y_k = a + bx_k + cx_k^2$, and

$$y - y_k = b(x - x_k) + c(x^2 - x_k^2), \quad \text{or} \quad \frac{y - y_k}{x - x_k} = (b + cx_k) + cx.$$

This last equation is of the first degree in x and $\frac{y - y_k}{x - x_k}$ so that the plot of $\left(x, \frac{y - y_k}{x - x_k}\right)$ will approximate a straight line.

(2) If the values of x are equidistant, *i.e.*, if they form an arithmetic progression, with common difference h, then if we replace x by $x + h$ in the equation, we get $y' = a + b(x + h) + c(x + h)^2$ and $\Delta y = y' - y = (bh + ch^2) + 2\,chx$. This last equation is of the first degree in x and Δy, so that the plot of $(x, \Delta y)$ will approximate a straight line.

Hence, *if a set of data may be approximately represented by the equation* $y = a + bx + cx^2$, *then* (1) *the plot of* $\left(x, \frac{y - y_k}{x - x_k}\right)$, *where* (x_k, y_k) *are the coördinates of any point on the experimental curve, will approximate a straight line, or* (2) *the plot of* $(x, \Delta y)$, *where the* Δy*'s are the differences in* y *formed for equidistant values of* x, *will approximate a straight line.*

The following examples will illustrate the method of determining the constants.

Example. In the following table, θ is the melting point in degrees Centigrade of an alloy of lead and zinc containing x per cent of lead. (From Saxelby's Practical Mathematics.)

x	θ	$x - 36.9$	$\theta - 181$	$\frac{\theta - 181}{x - 36.9}$	θ_c	Δ
87.5	292	50.6	111	2.20	295	−3
84.0	283	47.1	102	2.17	285	−2
77.8	270	40.9	89	2.18	268	+2
63.7	235	26.8	54	2.01	234	+1
46.7	197	9.8	16	1.63	199	−2
36.9	181	0	0		182	−1

In Fig. 77*a*, we have plotted (x, θ). We shall try to fit an equation of the form $\theta = a + bx + cx^2$ to the data. To verify this choice, observe that the curve passes through the point $x_k = 36.9$, $\theta_k = 181$, and plot the points $\left(x, \frac{\theta - 181}{x - 36.9}\right)$; this last plot approximates a straight line. (In plotting the ordinates for the straight line a scale unit ten times as large as that used for the ordinates of the experimental curve has been used; any further increase in the scale unit would simply magnify the devia-

tions.) We may now assume the relation $\frac{\theta - 181}{x - 36.9} = a' + b'x$, and use the method of averages to determine the constants. Dividing the data into two groups of three and two sets respectively and adding, we get

$$6.55 = 3a' + 249.3b',$$
$$3.64 = 2a' + 110.4b'.$$
$$\therefore \quad b' = 0.0130, \quad a' = 1.10.$$
$$\therefore \quad \frac{\theta - 181}{x - 36.9} = 1.10 + 0.0130x, \quad \text{or} \quad \theta = 141.4 + 0.620x + 0.0130x^2.$$

We now compute θ and the residuals.

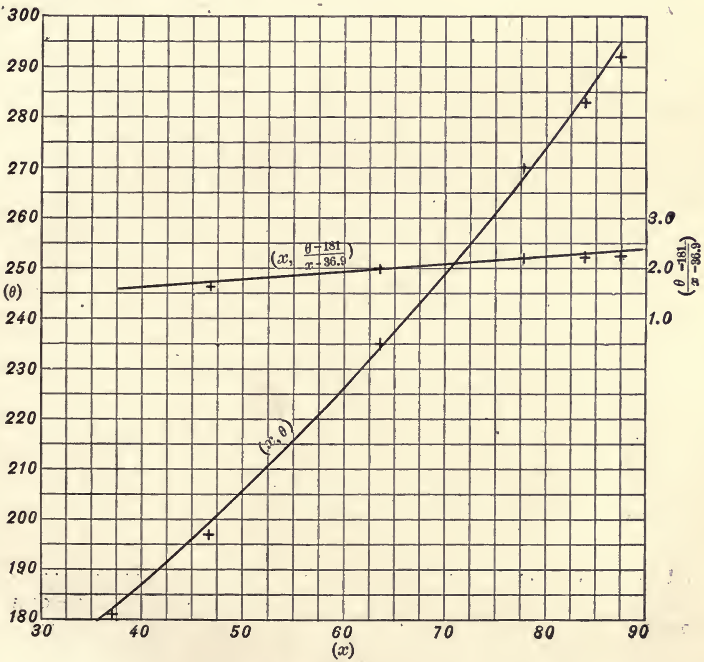


FIG. 77*a*.

Example. The following table gives the results of the measurements of train resistances; V is the velocity in miles per hour, R is the resistance in pounds per ton. (From Armstrong's Electric Traction.)

V	R	ΔR	V^2	R_c	Δ
20	5.5	3.6	400	5.70	−0.20
40	9.1	5.8	1,600	9.08	+0.02
60	14.9	7.9	3,600	14.82	+0.08
80	22.8	10.5	6,400	22.86	−0.06
100	33.3	12.7	10,000	33.22	+0.08
120	46.0		14,400	45.90	+0.10
Σ 420	131.6		36,400		

In Fig. 77*b*, the plot of (V, R) appears to be a parabola, $R = a + bV + cV^2$. Since the values of V are equidistant, we shall verify our choice of equation by a plot of $(V, \Delta R)$; this last plot approximates a straight line. We may therefore assume $\Delta R = (bh + ch^2) + 2\,ch\,V$, where $h = 20$.

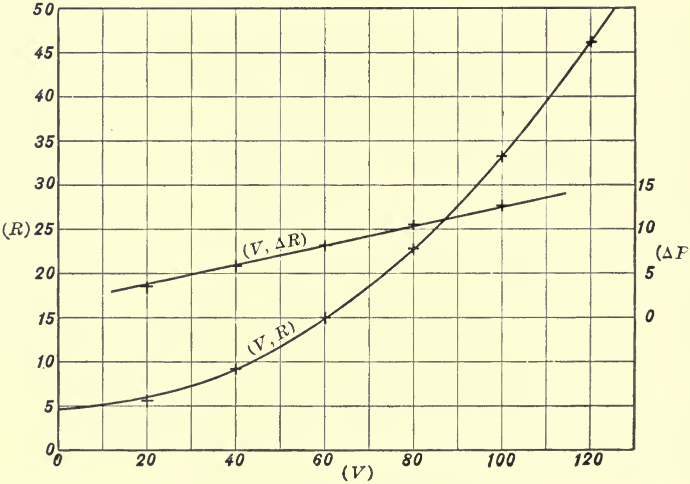


FIG. 77*b*.

We determine the constants in this last equation by the method of averages, using the five sets of values of V and ΔR. Dividing these data into two groups of three and two sets respectively and adding, we get

$$17.3 = 3\,(bh + ch^2) + 120\,(2\,ch),$$
$$23.2 = 2\,(bh + ch^2) + 180\,(2\,ch).$$

$$\therefore \quad 2\,ch = 0.117, \quad bh + ch^2 = 1.08.$$
$$\therefore \quad c = 0.0029, \quad b = -0.004.$$
$$\therefore \quad R = a - 0.004\,V + 0.0029\,V^2.$$

We determine a by substituting the six sets of values of V and R, and summing, thus

$$\Sigma R = 6a - 0.004\,\Sigma V + 0.0029\,\Sigma V^2,$$

or
$$131.6 = 6a - 0.004\,(420) + 0.0029\,(36{,}400),$$

and therefore
$$a = 4.62.$$

Hence, finally,
$$R = 4.62 - 0.004\,V + 0.0029\,V^2.$$

We now compute the values of R and the residuals; the agreement between the observed and calculated values of R is very close.

78. The hyperbola, $y = \frac{x}{a+bx} + c$. — This equation is a modification of the equation $y = \frac{x}{a+bx}$ discussed in Art. 74. In the latter equation, $x = 0$ gives $y = 0$, while in the former, $x = 0$ gives $y = c$. We may verify whether the equation $y = \frac{x}{a+bx} + c$ fits a set of experimental data as follows. Choose any point (x_k, y_k) on the experimental curve; then $y_k = \frac{x_k}{a+bx_k} + c$, and

$$y - y_k = \frac{a\,(x - x_k)}{(a+bx)\,(a+bx_k)}, \quad \text{or} \quad \frac{x - x_k}{y - y_k} = (a + bx_k) + \frac{b}{a}\,(a + bx_k)\,x.$$

This last equation is of the first degree in x and $\frac{x - x_k}{y - y_k}$, so that the plot of $\left(x, \frac{x - x_k}{y - y_k}\right)$ will approximate a straight line.

Hence, *if a set of data may be approximately represented by the equation $y = \frac{x}{a+bx} + c$, the plot of $\left(x, \frac{x - x_k}{y - y_k}\right)$, where (x_k, y_k) are the coördinates of a point on the experimental curve, will approximate a straight line.*

Example. The following table gives the results of experiments on the friction between a straw-fiber driver and an iron driven wheel under a pressure of 400 pounds; y is the coefficient of friction and x is the slip, per cent. (From Goss, Trans. Am. Soc. Mech. Eng., for 1907, p. 1099.)

x	y	$x - 0.65$	$y - 0.129$	$\frac{x - 0.65}{y - 0.129}$	y_c	y_c'
0.65	0.129	0	0		0.129	0.129
0.87	0.217	0.22	0.088	2.50	0.253	0.228
0.88	0.228	0.23	0.099	2.32	0.256	0.232
0.90	0.234	0.25	0.105	2.38	0.264	0.238
0.93	0.275	0.28	0.146	1.92	0.274	0.248
1.16	0.318	0.51	0.189	2.70	0.326	0.304
1.80	0.400	1.15	0.271	4.25	0.394	0.388
2.12	0.410	1.47	0.281	5.23	0.410	0.411
3.00	0.435	2.35	0.306	7.68	0.435	0.451

In Fig. 78 we have plotted the points (x, y); the experimental curve appears to be an hyperbola with an equation of the form $y = \frac{x}{a + bx} + c$. To verify this we note the point $x = 0.65$, $y = 0.129$ on the curve, and plot the points $\left(x, \frac{x - 0.65}{y - 0.129}\right)$. This last plot approximates a straight line. We may therefore assume the relation $\frac{x - 0.65}{y - 0.129} = a + bx$, and

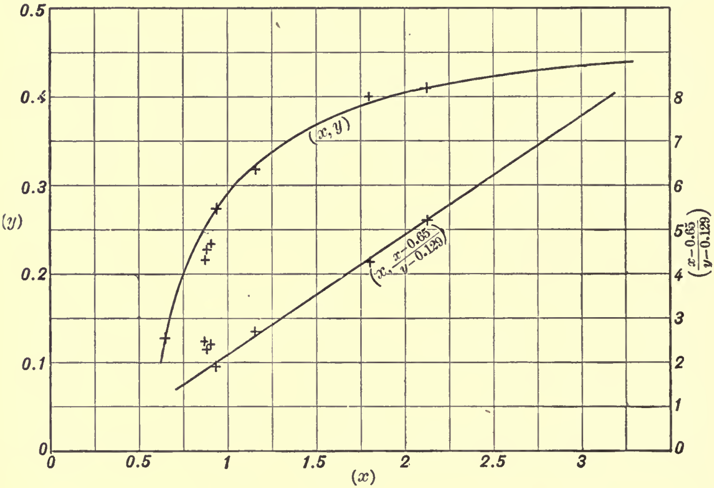


FIG. 78.

we shall determine the constants by the method of averages. As the first three points do not lie very near this straight line, we shall use only the last five sets of data, and dividing these into two groups of three and two sets respectively and adding, we get

$$8.87 = 3\,a + 3.89\,b,$$
$$12.91 = 2\,a + 5.12\,b.$$
$$\therefore \quad b = 2.77, \quad a = -0.64.$$

$$\therefore \quad \frac{x - 0.65}{y - 0\ 129} = -0.64 + 2.77\,x \quad \text{or} \quad y = \frac{x - 0.65}{2.77\,x - 0.64} + 0.129.$$

If we had used all eight points in determining the constants, we should have obtained

$$9.12 = 4\,a + 3.58\,b,$$
$$19.86 = 4\,a + 8.08\,b.$$
$$\therefore \quad b = 2.39, \quad a = 0.14.$$

$$\therefore \quad \frac{x - 0.65}{y' - 0.129} = 0.14 + 2.39\,x \quad \text{or} \quad y' = \frac{x - 0.65}{2.39\,x + 0.14} + 0.129.$$

We have computed both y and y' and note that the agreement with the observed values is probably as close as could be expected.

79. The logarithmic or exponential curve, $\log y = a + bx + cx^2$ or $y = ae^{bx+cx^2}$. — These equations are modifications of the logarithmic form $\log y = a + bx$ and the exponential form $y = ae^{bx}$. The equation $y = ae^{bx+cx^2}$ may be written $\log y = \log a + (b \log e)\, x + (c \log e)\, x^2$, and so is equivalent to the form $\log y = a + bx + cx^2$. This last equation is similar in form to the equation $y = a + bx + cx^2$ discussed in Art. 77, and the equation may be verified and the constants determined in a similar way.

Hence, *if a set of data may be approximately represented by the equation* $\log y = a + bx + cx^2$, *then* (1) *the plot of* $\left(x, \dfrac{\log y - \log y_k}{x - x_k}\right)$, *where* (x_k, y_k) *are the coördinates of a point on the experimental curve, will approximate a straight line, or* (2) *the plot of* $(x, \Delta \log y)$, *where the* $\Delta \log y$ *are the differences in* $\log y$ *formed for equidistant values of* x, *will approximate a straight line.*

Example. The following table gives the results of Winkelmann's experiments on the rate of cooling of a body in air; θ is the excess of temperature of the body over the temperature of its surroundings, t seconds from the beginning of the experiment.

t	θ	$\log \theta$	$\log \theta - \log 118.97$	$\dfrac{\log \theta - \log 118.97}{t}$	θ_c	Δ
0	118.97	2.07544	0		118.97	0
12.1	116.97	2.06808	−0.00736	−0.000608	116.99	−0.02
25.8	114.97	2.06059	−0.01485	−0.000576	114.97	0
41.7	112.97	2.05296	−0.02248	−0.000539	112.90	+0.07
59.7	110.97	2.04520	−0.03024	−0.000507	110.90	+0.07
82.0	108.97	2.03731	−0.03813	−0.000465	108.90	+0.07
109.0	106.97	2.02926	−0.04618	−0.000424	107.15	−0.18

In Fig. 79 we have plotted the points (t, θ). According to Newton's law of cooling, $\theta = ae^{bt}$ or $\log \theta = a + bt$, and so we have also plotted the points $(t, \log \theta)$; this last plot has a slight curvature. We shall therefore assume the law in the form $\log \theta = a + bt + ct^2$. To verify this, we note the point $t_k = 0$, $\theta_k = 118.97$ on the experimental curve, and plot the points $\left(t, \dfrac{\log \theta - \log 118.97}{t}\right)$; this plot approximates a straight line, so that we may assume $\dfrac{\log \theta - \log 118.97}{t} = b + ct$. We use the method of averages to determine the constants. Dividing the data into two groups of three sets each and adding, we get

$$-0.001723 = 3\,b + 79.6\, c,$$
$$-0.001396 = 3\,b + 250.7\, c.$$

$$\therefore \quad c = 0.000001911, \quad b = -0.000625.$$

$$\therefore \quad \frac{\log \theta - \log 118.97}{t} = -0.000625 + 0.000001911\,t$$

or
$$\log \theta = 2.07544 - 0.000625\,t + 0.000001911\,t^2.$$

We now compute θ and the residuals and note the close agreement between the observed and calculated values.

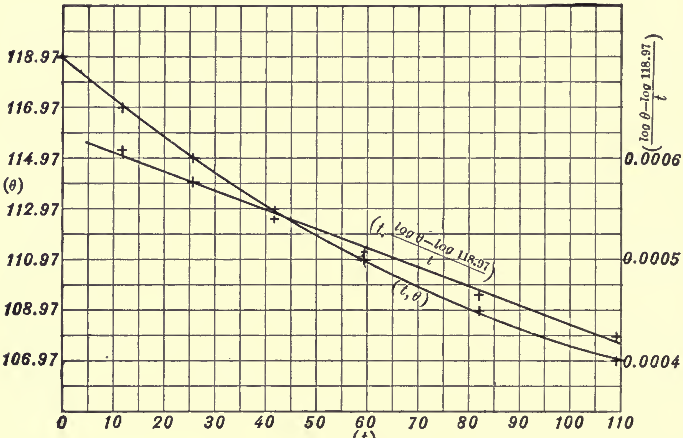


FIG. 79.

(IV) EQUATIONS INVOLVING FOUR OR MORE CONSTANTS.

80. The additional terms ce^{dx} and cx^d. — It is sometimes found that a simple equation will represent a part of our data very well and another part not at all, *i.e.*, the residuals $y_0 - y_c$ are very small for one part of our data and quite large for another part. Geometrically, this is equivalent to saying that the plot of the simple equation coincides approximately only with a part of the experimental curve. In such cases a modification of the simple equation by the addition of one or more terms will often cause the curves to fit approximately throughout. Such terms usually have the form ce^{dx} or cd^x, and added to our simple equations give the forms

$$y = a + bx + ce^{dx}, \qquad y = a + bx + cx^d,$$
$$y = ae^{bx} + ce^{dx}, \qquad y = ax^b + cx^d,$$
$$y = \frac{x}{a + bx} + ce^{dx}, \qquad y = \frac{x}{a + bx} + cx^d, \quad \text{etc.}$$

We shall give a few examples to illustrate some of these cases.

81. The equation $y = a + bx + ce^{dx}$.—If a part of the experimental curve approximates a straight line, we may fit an equation of the form $y = a + bx$ to this part of the curve. The deviation of this straight line from the remainder of the experimental curve (Fig. 81*a*) will be measured by the residuals $r = y_0 - y_c = y - (a + bx)$. We now plot (x, r) and study the nature of this plot. We may be able to represent this plot by means of the simple exponential $r = ce^{dx}$, where the values of the constants c and d are such that the value of r is negligible for that part of the plot to which the straight line has been fitted. The entire experimental curve can thus be represented by $ce^{dx} = y - (a + bx)$ or $y = a + bx + ce^{dx}$.

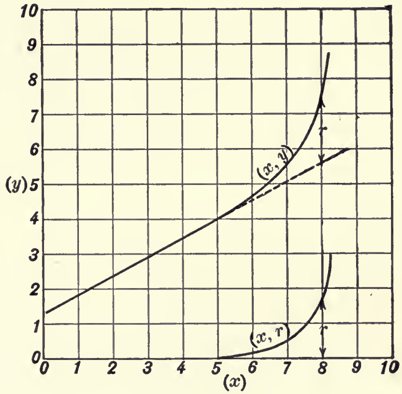


FIG. 81*a*.

The equation $y = a + bx + ce^{dx}$ may fit an experimental curve although no part of the curve is approximately a straight line; this means that the values of the term ce^{dx} are not negligible for any values of x. If the values of x are equidistant, we may verify that this equation is the correct one to assume by the following method. Let the constant difference in the values of x be h. If we replace x by $x + h$, we get

$$y' = a + b(x + h) + ce^{d(x+h)},$$

and, therefore, for the difference in the values of y,

$$\Delta y = y' - y = bh + ce^{dx}(e^{dh} - 1).$$

If Δy and $\Delta y'$ are two successive values of Δy, then

$$\Delta y' = bh + ce^{d(x+h)}(e^{dh} - 1),$$

and the difference in the values of Δy is

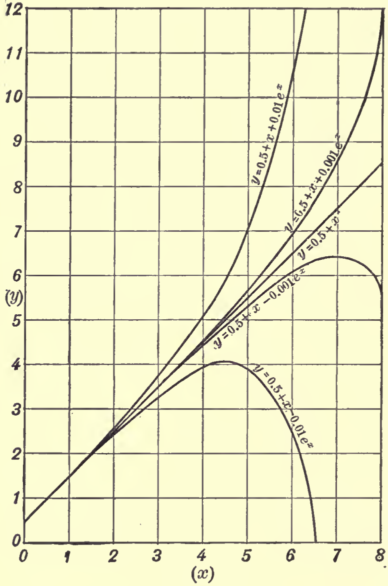
$$\Delta^2 y = \Delta y' - \Delta y = ce^{dx}(e^{dh} - 1)^2.$$

Hence,

$$\log \Delta^2 y = \log c\,(e^{dh} - 1)^2 + (d \log e)\,x.$$

The last equation is of the first degree in x and $\log \Delta^2 y$ so that the plot of $(x, \log \Delta^2 y)$ will approximate a straight line. From this straight

line we may determine the constants $\log c\,(e^{dh} - 1)^2$ and $d \log e$ and therefore c and d in the usual way. We now write the equation in the form $y - ce^{dx} = a + bx$, and from the straight line plot of $(x,\ y - ce^{dx})$, we determine the constants a and b.



$y = a + bx + ce^{dx}$

FIG. 81b.

In Fig. 81b we have plotted the equations

$$y = 0.5 + x,$$
$$y = 0.5 + x - 0.01\,e^x,$$
$$y = 0.5 + x - 0.001\,e^x,$$
$$y = 0.5 + x + 0.01\,e^x,$$
$$y = 0.5 + x + 0.001\,e^x.$$

Example. The following data are the results of experiments made with a gasometer by means of which the amount of air which passes into a receiving tank can be measured; x is the vacuum in the tank in inches of mercury, y is the number of cu. ft. of air per minute passing into the tank. (Experiments made by W. D. Canan at the Mass. Inst. of Tech.)

x	y	y'	$r = y' - y$	$\log r$	r_c	y_c	Δ
8	1.17	1.49	0.32	9.5051 − 10	0.322	1.17	0
10	1.37	1.55	0.18	9.2553 − 10	0.179	1.37	0
12	1.50	1.61	0.11	9.0414 − 10	0.099	1.51	−0.01
14	1.62	1.67	0.05	8.6990 − 10	0.055	1.61	+0.01
16	1.71	1.73	0.02		0.031	1.70	+0.01
18	1.80	1.79	−0.01		0.017	1.77	+0.03
20	1.85	1.85	0		0.009	1.84	+0.01
22	1.91	1.91	0		0.005	1.90	+0.01
24	1.96	1.97	0.01		0.003	1.97	−0.01
26	2.02	2.03	0.01		0.002	2.03	−0.01
28	2.10	2.09	−0.01		0.001	2.09	+0.01

In Fig. 81c we note that the plot of (x, y) approximates a straight line for values of $x > 14$, and we shall fit an equation of the form

$y' = a + bx$ to this part of the data. Using the method of averages and dividing the data into two groups of four and three sets, we have

$$7.27 = 4a + 76b,$$
$$6.08 = 3a + 78b,$$
$$\therefore \quad b = 0.03, \quad a = 1.25$$

and
$$y' = 1.25 + 0.03x.$$

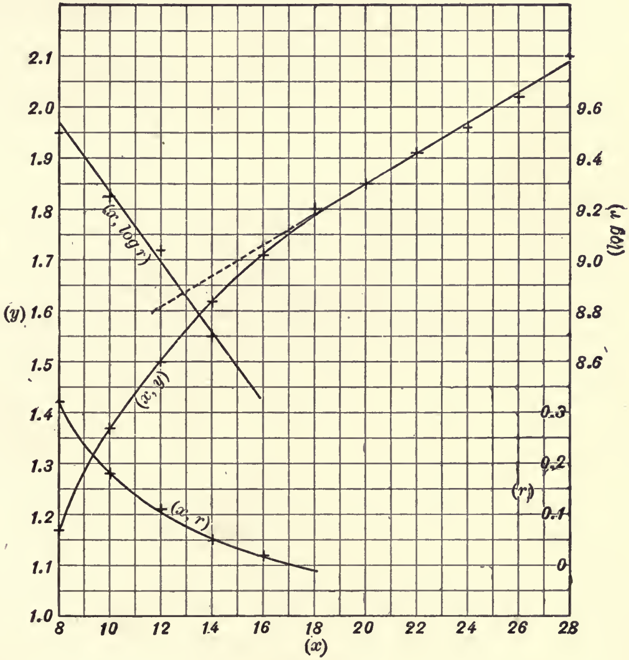


FIG. 81c.

Now compute the values of y' and the residuals $r = y' - y$ (by taking $r = y' - y$ instead of $r = y - y'$, the residuals are positive and easier to handle in the subsequent calculations). Plot (x, r) for values of $x < 14$ and study the nature of this plot; this seems to be a simple exponential, $r = ce^{dx}$; verify this by plotting $(x, \log r)$ and note that this plot approximates a straight line. Using the method of averages determine the constants in the equation $\log r = \log c + (d \log e)\, x$; thus

$$8.7604 - 10 = 2 \log c + 18\, d \log e,$$
$$7.7404 - 10 = 2 \log c + 26\, d \log e.$$

$\therefore \quad d \log e = 9.8725 - 10 = -0.1275, \quad \log c = 0.5277.$

$\therefore \quad d = -0.294, \quad c = 3.37.$

$\therefore \quad \log r = 0.5277 - 0.1275\,x, \quad$ and $\quad r = 3.37\,e^{-0.294\,x}$

The final equation is

$$y = 1.25 + 0.03\,x - 3.37\,e^{-0.294\,x}.$$

Now compute y and the residuals, and note the close agreement between the observed and calculated values.

82. The equation $y = ae^{bx} + ce^{dx}$. — A part of the experimental curve may be represented by a simple exponential $y = ae^{bx}$, *i.e.*, a part of the plot of $(x, \log y)$ approximates a straight line. We then study the deviations, $r = y_0 - y_c = y - ae^{bx}$, of this exponential curve from the rest of the experimental curve. The plot of (x, r) may be representable by another exponential, $r = ce^{dx}$, where the values of r are negligible for that part of the experimental curve to which $y = ae^{bx}$ has been fitted. The entire curve can then be represented by the equation $y = ae^{bx} + ce^{dx}$.

The equation $y = ae^{bx} + ce^{dx}$ may fit an experimental curve although no part of the curve can be approximated by the simple exponential $y = ae^{bx}$. If the values of x are equidistant, we may verify that this equation is the correct one to assume by the following method. Let the constant difference in the values of x be h. Consider three successive values x, $x + h$, $x + 2\,h$ and their corresponding values y, y', y''. We evidently have

$$\begin{aligned} y &= ae^{bx} + ce^{dx}, \\ y' &= ae^{b\,(x+h)} + ce^{d\,(x+h)} = ae^{bx}e^{bh} + ce^{dx}e^{dh}, \\ y'' &= ae^{b\,(x+2\,h)} + ce^{d\,(x+2\,h)} = ae^{bx}e^{2\,bh} + ce^{dx}e^{2\,dh}. \end{aligned}$$

Now eliminate e^{bx} and e^{dx} from these three equations by multiplying the first equation by $e^{(b+d)h}$, the second by $-(e^{bh} + e^{dh})$, and adding the results to the third equation. We get

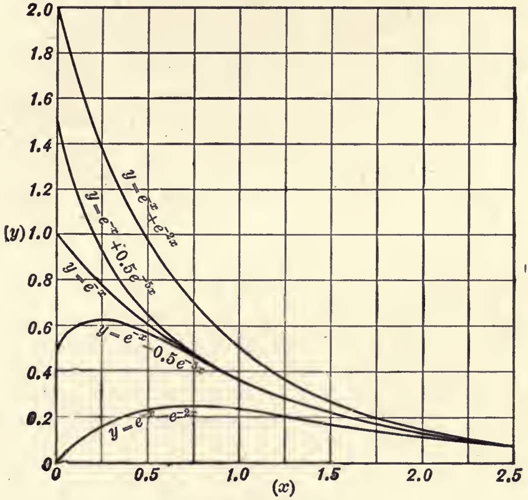
$$y'' - (e^{bh} + e^{dh})\,y' + e^{(b+d)\,h}y = 0,$$

or

$$\frac{y''}{y} = (e^{bh} + e^{dh})\,\frac{y'}{y} - e^{(b+d)\,h}$$

This is an equation of the first degree in y'/y and y''/y so that the plot of $(y'/y, y''/y)$ will approximate a straight line. From this straight line determine the constants $e^{bh} + e^{dh}$ and $e^{(b+d)h}$, and hence b and d as usual. We now write the original equation $ye^{-dx} = ae^{(b-d)\,x} + c$. This is a linear equation in $e^{(b-d)\,x}$ and ye^{-dx} so that the plot of $(e^{(b-d)\,x}, ye^{-dx})$ would approximate a straight line. From this straight line determine the values of the constants a and c.

In Fig. 82*a*, we have plotted the equations $y = e^{-x}$, $y = e^{-x} + 0.5\,e^{-5x}$, $y = e^{-x} - 0.5\,e^{-5x}$, $y = e^{-x} + e^{-2x}$, $y = e^{-x} - e^{-2x}$.



$y = ae^{bx} + ce^{dx}$

FIG. 82*a*.

Example. The following are the measurements made on a curve recorded by an oscillograph representing a change of current i due to a change in the conditions of an electric circuit t. (From Steinmetz, Engineering Mathematics.)

t	i	$\log i$	i'	$r = i' - i$	$\log r$	r_c	i_c	Δ
0	2.10	0.3222	4.94	2.84	0.4533	2.85	2.09	+0.01
0.1	2.48	0.3945	4.44	1.96	0.2923	1.96	2.48	0
0.2	2.66	0.4249	3.99	1.33	0.1239	1.34	2.65	+0.01
0.4	2.58	0.4116	3.22	0.64	9.8062 − 10	0.63	2.59	−0.01
0.8	2.00	0.3010	2.10	0.10	9.0000 − 10	0.14	1.96	+0.04
1.2	1.36	0.1335	1.37	0.01		0.03	1.34	+0.02
1.6	0.90	9.9542 − 10	0.89	−0.01		0.01	0.88	+0.02
2.0	0.58	9.7634 − 10	0.58	0		0	0.58	0
2.5	0.34	9.5315 − 10	0.34	0		0	0.34	0
3.0	0.20	9.3010 − 10	0.20	0		0	0.20	0

In Fig. 82*b* we note that the right-hand part of the plot of (t, i) appears to be exponential. We verify the choice of $i' = ae^{bt}$ by plotting $(t, \log i)$ and noting that this plot approximates a straight line for values of

$t > 0.8$. We therefore assume $\log i' = \log a + (b \log e)\, t$, and using the method of averages for the values of $t > 0.8$, we have

$$9.8511 - 10 = 3 \log a + 4.8\, b \log e,$$
$$8.8325 - 10 = 2 \log a + 5.5\, b \log e.$$

$\therefore \quad b \log e = 9.5356 - 10 = -0.4644, \quad \log a = 0.6934,$

$\therefore \quad b = -1.07, \quad a = 4.94,$

and $\qquad \log i' = 0.6934 - 0.4644\, t, \quad$ or $\quad i' = 4.94\, e^{-1.07\,t}.$

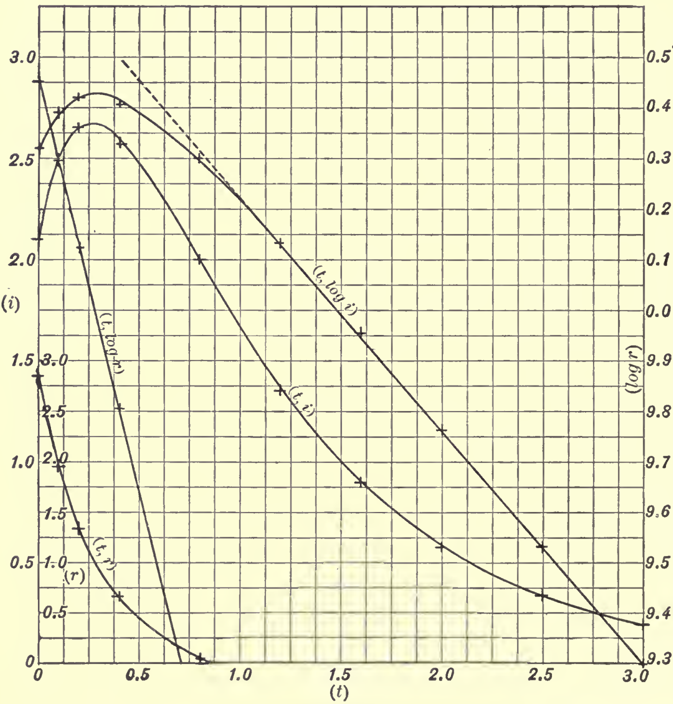


FIG. 82*b*.

Now find the values of i' and the residuals $r = i' - i$; these residuals are practically negligible for values of $t > 0.8$. We plot (t, r) and try to fit an equation to this curve. This again appears to be exponential and we verify this by plotting $(t, \log r)$; the plot approximates a

straight line, except for $t = 0.8$. We therefore assume $r = ce^{dt}$ or $\log r = \log c + (d \log e)\, t$. Using the method of averages for $t < 0.8$, we have

$$0.7456 = 2 \log c + 0.1\, d \log e,$$
$$9.9301 - 10 = 2 \log c + 0.6\, d \log e.$$
$$\therefore \quad d \log e = -1.6310, \quad \log c = 0.4544.$$
$$\therefore \; d = -3.76, \quad c = 2.85,$$

and $\log r = 0.4544 - 1.6310\, t$, or $r = 2.85\, e^{-3.76 t}$.
The final equation is

$$i = 4.94\, e^{-1.07 t} - 2.85\, e^{-3.76 t}.$$

We now compute i and the residuals and note the very close agreement between the observed and computed values of i.

83. The polynomial $y = a + bx + cx^2 + dx^3 + \cdots$. — The equation $y = a + bx + cx^2$ may be modified by the addition of another term into $y = a + bx + cx^2 + dx^3$. If the values of x are equidistant, we may verify the correctness of the assumption of the last equation by the following method. Let the constant difference in the values of x be h. Then the successive differences in the values of y are

$$\Delta y = (bh + ch^2 + dh^3) + (2\, ch + 3\, dh^2)\, x + 3\, dhx^2,$$
$$\Delta^2 y = (2\, ch^2 + 6\, dh^3) + 6\, dh^2 x,$$
$$\Delta^3 y = 6\, dh^3.$$

Hence the plot of $(x, \Delta^2 y)$ will approximate a straight line, and the values of $\Delta^3 y$ are approximately constant. From the equation of the straight line we may determine the constants c and d, and writing the original equation in the form $(y - cx^2 - dx^3) = a + bx$, the plot of $(x, y - cx^2 - dx^3)$ will approximate a straight line, from which the constants a and b may be determined. Another method of determining the constants a, b, c, d in the equation $y = a + bx + cx^2 + dx^3$ consists in selecting four points on the experimental curve, substituting their coördinates in the equation, and solving the four linear equations thus obtained for the values of the four quantities a, b, c, and d.

In a similar manner the polynomial $y = a + bx + cx^2 + \cdots + kx^n$ may be determined so that the corresponding curve passes through $n + 1$ points of the experimental curve; it is simply necessary to substitute the coördinates of these $n + 1$ points in the equation and to solve the $n + 1$ linear equations for the values of the $n + 1$ quantities, a, b, c, . . . , k. If the values of x are equidistant, we can show that the plot of $(x, \Delta^{n-1} y)$ is a straight line and that $\Delta^n y$ is constant, where $\Delta^{n-1} y$ and $\Delta^n y$ are the $(n - 1)$st and nth order of differences in the values of y. Thus, if a sufficient number of terms are taken in the equation of the polynomial, this polynomial may be made to represent any set of data exactly; but it is not wise to force a fit in this way, since the determination of a large number of constants is very laborious, and in many

cases a much simpler equation involving fewer constants may give much more accurate results in subsequent calculations.

We shall work a single example to illustrate the method of determining the constants.

Example. We wish to fit a polynomial equation to the following data:

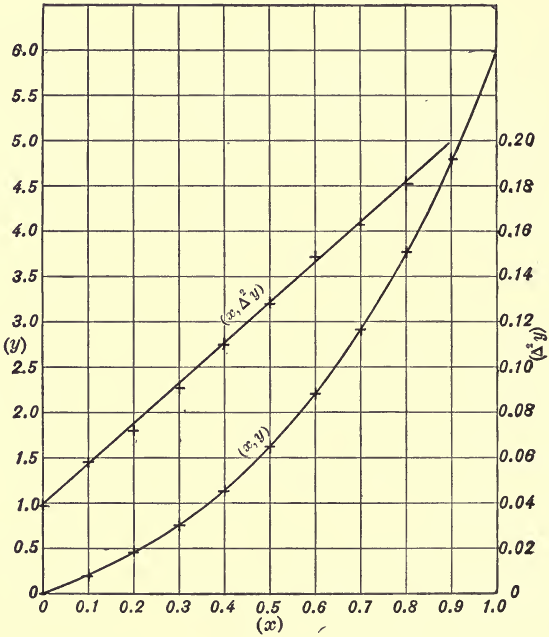


FIG. 83.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	y_c	Δ
0	0	0.212	0.039	0.019	0	0
0.1	0.212	0.251	0.058	0.014	0.210	+0.002
0.2	0.463	0.309	0.072	0.019	0.463	0
0.3	0.772	0.381	0.091	0.019	0.770	+0.002
0.4	1.153	0.472	0.110	0.018	1.152	+0.001
0.5	1.625	0.582	0.128	0.021	1.625	0
0.6	2.207	0.710	0.149	0.014	2.209	−0.002
0.7	2.917	0.859	0.163	0.018	2.920	−0.003
0.8	3.776	1.022	0.181		3.776	0
0.9	4.798	1.203			4.797	+0.001
1.0	6.001				5.998	+0.003

In Fig. 83 we have plotted (x, y). We form the successive differences and note that the third differences are approximately constant, and that the plot of $(x, \Delta^2 y)$ approximates a straight line (Fig. 83). We may therefore assume an equation of the form $y = a + bx + cx^2 + dx^3$, or $y = bx + cx^2 + dx^3$, since the curve evidently passes through the origin of coördinates. To determine the constants b, c, and d, select three points on the experimental curve; three such points are (0.2, 0.463), (0.5, 1.625), and (0.8, 3.776). Substituting these coördinates in the equation, we get

$$\begin{aligned} 0.463 &= 0.2\,b + 0.04\,c + 0.008\,d, \\ 1.625 &= 0.5\,b + 0.25\,c + 0.125\,d, \\ 3.776 &= 0.8\,b + 0.64\,c + 0.512\,d. \end{aligned}$$

Solving these equations for b, c, and d, we have

$$b = 1.989, \quad c = 1.037, \quad d = 2.972$$

and hence the equation is

$$y = 1.989\,x + 1.037\,x^2 + 2.972\,x^3.$$

We now compute the values of y and the residuals.

84. Two or more equations. — It is sometimes impossible to represent a set of data by a simple equation involving few constants or even by a complex equation involving many constants. In such cases it is often convenient to represent a part of the data by one equation and another part of the data by another equation. The entire set of data will then be represented by two equations, each equation being valid for a restricted range of the variables. Thus, Regnault represented the relation between the vapor pressure and the temperature of water by three equations, one for the range from $-32°$ F. to $0°$ F., another for the range from $0°$ F. to $100°$ F., and a third for the range from $100°$ F. to $230°$ F. Later, Rankine, Marks, and others represented the relation by a single equation. The following example will illustrate the representation of a set of data by two simple equations.

Example. The following data are the results of experiments on the collapsing pressure, p in pounds per sq. in. of Bessemer steel lap-welded tubes, where d is the outside diameter of the tube in inches and t is the thickness of the wall in inches. (Experiments reported by R. T. Stewart in the Trans. Am. Soc. of Mech. Eng., Vol. XXVII, p. 730.)

$\frac{t}{d}$	P	$\log \frac{t}{d}$	$\log P$	P_c	Δ
0.0165	225	8.2175 − 10	2.3522	230	−5
0.0194	383	8.2878 − 10	2.5832	381	+2
0.0216	524	8.3345 − 10	2.7193	533	−9
0.0214	536	8.3304 − 10	2.7292	517	+19

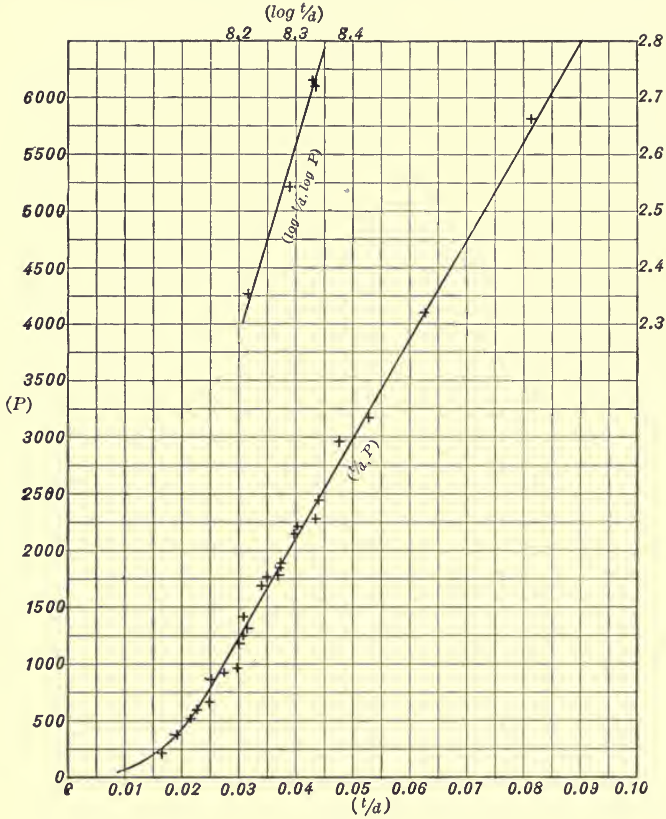


FIG. 84.

$\frac{t}{d}$	P	P_c	Δ	$\frac{t}{d}$	P	P_c	Δ
0.0228	592	570	+ 22	0.0370	1779	1821	− 42
0.0250	670	764	− 84	0.0374	1860	1856	+ 4
0.0253	870	790	+ 80	0.0375	1879	1865	+ 14
0.0277	928	1002	− 74	0.0400	2147	2085	+ 62
0.0298	964	1187	−223	0.0403	2224	2112	+112
0.0299	1184	1196	− 12	0.0436	2280	2403	−123
0.0309	1251	1284	− 33	0.0442	2441	2455	− 14
0.0316	1319	1346	− 27	0.0477	2962	2764	+198
0.0309	1419	1284	+135	0.0527	3170	3204	− 34
0.0343	1680	1583	+ 97	0.0628	4095	4194	− 99
0.0349	1762	1636	+126	0.0815	5560	5741	−181

It should be noted that a set of corresponding values of t/d and P are not the results of a single experiment but the averages of groups containing from two to twenty experiments.

Following the work of Prof. Stewart, we have plotted $(t/d, P)$, Fig. 84, and note that the experimental curve approximates a straight line for all values of t/d except the first four, *i.e.*, for values of $t/d > 0.023$. We may therefore assume $P = a + b\left(\frac{t}{d}\right)$. If we use the method of selected points to determine the constants a and b we may choose the points $t/d = 0.065$, $P = 4250$, and $t/d = 0.030$, $P = 1215$ as lying on the straight line; we then have

$$4250 = a + 0.065\,b,$$
$$1215 = a + 0.030\,b.$$

$$\therefore \quad b = 86{,}714, \quad a = -1386$$

and
$$P = 86{,}714\left(\frac{t}{d}\right) - 1386.$$

This result agrees with that given by Prof. Stewart. If we use the method of averages to determine the constants a and b we divide the last 22 sets of data into two groups of 11 each, and get

$$12{,}639 = 11\,a + 0.3231\,b,$$
$$30{,}397 = 11\,a + 0.5247\,b.$$

$$\therefore \quad b = 88{,}085, \quad a = -1438,$$

and
$$P = 88{,}055\left(\frac{t}{d}\right) - 1438.$$

In our table we have given the values of P computed from this last formula. The values of P computed from the first formula agree very closely with these. It is seen that the percentage deviations are in general quite small though large in a few cases, varying from 0.2 per cent to 10 per cent, which is to be expected from the nature of the experiments.

We now attempt to fit an equation to the first four sets of data. The addition of a modifying term of the form $c\left(\frac{t}{d}\right)^k$ or $ce^{k\frac{t}{d}}$ to the above formula is not successful here. We shall therefore follow Prof. Stewart's work and attempt to fit an equation of the parabolic form, $P = a\left(\frac{t}{d}\right)^b$. We verify this choice by plotting $\left(\log\frac{t}{d}, \log P\right)$ and observing that this plot approximates a straight line. (The fewness of the experiments for values of $t/d < 0.023$ is a handicap here.) Assuming

$$\log P = \log a + b\log\left(\frac{t}{d}\right),$$

and using the method of averages, we find

$$4.9354 = 2 \log a + (6.5053 - 10)\, b,$$
$$5.4485 = 2 \log a + (6.6649 - 10)\, b.$$
$$\therefore \quad b = 3.11, \quad a = 80{,}580{,}000$$

and
$$P = 80{,}580{,}000 \left(\frac{t}{d}\right)^{3.11}.$$

We compute the values of P from this formula.

The entire set of data have thus been represented by means of two simple equations, each valid for a restricted range of the variables.*

EXERCISES.

[*Note.* The exercises which follow are divided into two sets. The type of equation that will approximately represent the empirical data is suggested for each example in the first set. For the examples in the second set, the choice of a suitable equation is left to the student.]

1. Temperature coefficient; r is the resistance of a coil of wire in ohms, θ is the temperature of the coil in degrees Centigrade. [$y = a + bx$]

r	10.421	10.939	11.321	11.799	12.242	12.668
θ	10.50	29.49	42.70	60.01	75.51	91.05

2. Galvanometer deflection; D is the deflection in mm., I is the current in micro-amperes. [$y = a + bx$]

D	29.1	48.2	72.7	92.0	118	140	165	199
I	0.0493	0.0821	0.123	0.154	0.197	0.234	0.274	0.328

3. Volt-ampere characteristic of 118 volt tungsten lamp; e is the terminal voltage, i is the current. [$y = ax^b$]

e	2	4	8	16	25	32	50	64	100	125
i	0.0245	0.0370	0.0570	0.0855	0.1125	0.1295	0.1715	0.2000	0.2605	0.2965

e	150	180	200	218
i	0.3295	0.3635	0.3865	0.4070

4. Pressure-volume of saturated steam; v is the volume in cu. ft. of 1 pound of steam, p is the pressure in pounds per sq. in. [$y = ax^b$]

v	26.43	22.40	19.08	16.32	14.04	12.12	10.51	9.147	7.995
p	14.70	17.53	20.80	24.54	28.83	33.71	39.25	45.49	52.52

5. Chemical concentration experiment; x is the concentration of hydrogen ions, y is the concentration of undissociated hydrochloric acid. [$y = ax^b$]

x	1.68	1.22	0.784	0.426	0.092	0.047	0.0096	0.0049	0.00098
y	1.32	0.676	0.216	0.074	0.0085	0.00315	0.00036	0.00014	0.000018

6. Vibration of a long pendulum; A is the amplitude in inches, t is the time since it was set swinging. [$y = ae^{bx}$]

t	0	1	2	3	4	5	6
A	10	4.97	2.47	1.22	0.61	0.30	0.14

* Prof. Peddle in "The Construction of Graphical Charts" has fitted the equation $t/d = 0.00274 \sqrt[3]{P} + 0.0000000011\, P^2$ to Prof. Stewart's data.

7. Newton's law of cooling; θ is the excess of the temperature of the body over the temperature of its surroundings, t is the time in seconds since the beginning of the experiment. $[y = ae^{bx}]$

t	0	3.45	10.85	19.30	28.80	40.10	53.75	70.95
θ	19.9	18.9	16.9	14.9	12.9	10.9	8.9	6.9

8. Barometric pressure; p is the pressure in inches of mercury, h is the height in ft. above sea level. $[y = ae^{bx}]$

h	0	886	2753	4763	6942	10,593
p	30	29	27	25	23	20

9. Electric arc of length 4 mm.; V is the potential difference in volts, i is the current in amperes. $\left[y = a + \frac{b}{x}\right]$

i	2.46	2.97	3.45	3.96	4.97	5.97	6.97	7.97
V	67.7	65.0	63.0	61.0	58.25	56.25	55.10	54.30

10. Speed of a vessel; $H.P.$ is the horse power developed, v is the speed in knots. $[y = a + bx^3]$

v	5	7	9	11	12
$H.P.$	290	560	1144	1810	2300

11. Hydraulic transmission; $H.P.$ is the horsepower supplied at one end of a line of pipes, u is the useful power delivered at the other end. $\left[\frac{y}{x} = a + bx^2\right]$

$H.P.$	100	150	200	250	300
u	96.5	138	172	196	206

12. Magnetic characteristic of iron; H is the number of gilberts per cm., a measure of the field intensity, B is the number of kilolines per sq. cm., a measure of the flux density. $\left[y = \frac{x}{a + bx}\right]$

H	8	10	15	20	30	40	60	80
B	13.0	14.0	15.4	16.3	17.2	17.8	18.5	18.8

13. Focal distance of a lens; p is the distance of the object, p' is the distance of its image. $\left[y = \frac{x}{a + bx}\right]$

p	320	240	180	140	120	100	80	60
p'	21.35	21.80	22.50	23.20	23.80	24.60	26.20	29.00

14. Pressure-volume in a gas engine; p is the pressure in pounds per sq. in., v is the volume in cu. ft. per pound. $[y = ax^b + c]$

p	44.7	53.8	73.5	85.8	113.2	135.8
v	7.03	5.85	4.30	3.50	2.50	1.90

15. Law of cooling; θ is the temperature of a vessel of cooling water, t is the time in minutes since the beginning of observation. $[y = ae^{bx} + c]$

t	0	1	2	3	5	7	10	15	20
θ	92.0	85.3	79.5	74.5	67.0	60.5	53.5	45.0	39.5

16. Straw-fibre friction at 150 pounds pressure according to Goss's experiments; y is the coefficient of friction for a straw-fibre driver and an iron driven wheel, x is the slip, per cent. $\left[y = \dfrac{x}{a + bx} + c\right]$

x	0.153	0.179	0.213	0.271	0.313	0.359	0.368	0.381	0.386	0.405
y	0.56	0.58	0.61	0.78	0.99	1.10	1.04	1.22	1.40	1.75

x	0.411	0.432	0.458	0.463	0.465	0.473
y	1.94	2.00	2.25	2.33	3.15	2.79

17. Expansion of mercury according to Regnault's experiments; γ is the coefficient of expansion between 0° C. and t° C. $[y = a + bx + cx^2]$

t	0	100	150	200	250	300	360
γ	0.00018179	0.00018216	0.00018261	0.00018323	0.00018403	0.00018500	0.00018641

18. Velocity of water in Mississippi River; v is the velocity, D is the depth. $[y = a + bx + cx^2]$

D	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
v	3.1950	3.2299	3.2532	3.2611	3.2516	3.2282	3.1807	3.1266	3.0594	2.9759

19. Solution of potassium chromate; s is the weight of potassium chromate which will dissolve in 100 parts by weight of water at a temperature of t° C. $[\log y = a + bx + cx^2]$

t	0	10	27.4	42.1
s	61.5	62.1	66.3	70.3

20. Load-elongation of annealed high carbon steel wire of diameter 0.0693 and gage length 30 in.; W is the load in pounds, E is the elongation in inches. $[y = a + bx + ce^{dx}]$

W	0	50	100	150	200	225	250	260	280	290	300	310
E	0	0.0130	0.0251	0.0387	0.0520	0.0589	0.0659	0.0689	0.0746	0.0778	0.0807	0.0842

W	320	330	340	350	360
E	0.0877	0.0916	0.0980	0.1111	0.1420

21. Load-elongation of wire of Ex. 20 in hard-drawn condition; W is the load in pounds, E is the elongation in inches. $[y = a + bx + cx^d]$

W	0	100	200	300	400	500	600	700	800	850	900
E	0	0.0280	0.0562	0.0849	0.1150	0.1471	0.1820	0.2191	0.2628	0.2879	0.3166

22. Empirical curve. $[y = ae^{bx} + ce^{dx}]$

x	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	3.0
y	3.00	1.89	1.27	0.88	0.63	0.46	0.33	0.25	0.18	0.10

23. Magnetic characteristic of iron; H is the number of gilberts per cm., a measure of the field intensity, B is the number of kilolines per sq. cm., a measure of the flux density (cf. Ex. 12). $\left[y = \dfrac{x}{a + bx} + ce^{dx}\right]$

H	2	4	6	8	10	15	20	30	40	60	80
B	3.0	8.4	11.2	13.0	14.0	15.4	16.3	17.2	17.8	18.5	18.8

24. Speed of a vessel; I is the indicated horsepower, v is the speed in knots. $[y = a + bx + cx^2 + dx^3]$

v	8	9	10	11	12	13	14	15	16	17	18
I	1000	1400	1900	2500	3250	4200	5400	6950	8950	11,450	15,400

25. Test on square steel wire for winding guns; S is the stress in pounds per sq. in., E is the elongation in inches per inch.

S	5000	10,000	20,000	30,000	40,000	50,000	60,000	70,000	80,000
E	0	0.00019	0.00057	0.00094	0.00134	0.00173	0.00216	0.00256	0.00297

S	90,000	100,000	110,000
E	0.00343	0.00390	0.00444

26. Flow of water over a Thomson gauge notch; Q is the number of cu. ft. of water, H is the head in feet.

H	1.2	1.4	1.6	1.8	2.0	2.4
Q	4.2	6.1	8.5	11.5	14.9	23.5

27. Friction between belt and pulley; θ is the arc of contact in radians between belt and pulley, P is the pull in pounds applied to one end of pulley to raise a weight W at the other end.

θ	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
P	5.62	6.93	8.52	10.50	12.90	15.96	19.67	24.24	29.94

28. Electric arc of length 2 mm.; V is the potential difference in volts, i is the current in amperes.

i	1.96	2.46	2.97	3.45	3.96	4.97	5.97	6.97	7.97	9.00
V	50.25	48.70	47.90	47.50	46.80	45.70	45.00	44.00	43.60	43.50

29. Normal induction curve for transformer steel; H is the number of gilberts per cm., B is the number of lines per sq. cm.

H	1.0	1.3	2.1	2.9	3.4	4.1	4.5	5.2	5.9	7.5	9.0	11.0
B	425	800	1750	2850	4300	6100	6725	7800	8600	10,200	11,150	12,200

30. Pressure-volume in a gas engine; p is the pressure in pounds per sq. in., v is the volume in cu. ft. per pound.

p	39.6	44.7	53.8	73.5	85.8	113.2	135.8	178.2
v	10.61	9.73	8.55	7.00	6.23	5.18	4.59	3.87

31. Melting point of alloy of lead and zinc; θ is the temperature in degrees Centigrade, x is % of lead.

x	40	50	60	70	80	90
θ	186	205	226	250	276	304

32. Empirical curve.

x	1	3	5	7	9	11	13
y	6.42	8.50	11.03	14.03	17.53	21.55	26.12

33. Candle-power of an incandescent lamp; H is the age of the lamp in hours, $C.P.$ is the candle-power.

H	0	250	500	750	1000	1250	1500
$C.P.$	24.0	17.6	16.5	15.8	15.3	14.9	14.5

34. Insulation resistance-current passes through insulator and galvanometer; D is the deflection of the galvanometer, t is the time in minutes.

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
D	18	11	8.0	6.2	5.5	5.0	4.4	4.0	3.5	3.3	3.0	2.7	2.5	2.5	2.4

35. Experiments with a crane; f is the force in pounds which will just overcome a weight w.

w	100	200	300	400	500	600	700	800
f	8.5	12.8	17.0	21.4	25.6	29.9	34.2	38.5

36. Copper-nickel thermocouple; t is the temperature in degrees, p is the thermoelectric power in microvolts.

t	0	50	100	150	200
p	24	25	26	26.9	27.5

37. Law of falling body; s is the distance in cm. fallen by body in t sec.

t	0.2477	0.4175	0.5533	0.6760	0.7477
s	30.13	85.26	150.39	223.60	274.20

38. Loads which cause the failure of long wrought-iron columns with rounded ends; P/a is the load in pounds per sq. in., l/r is the ratio of length of column to the least radius of gyration of its cross-section.

l/r	140	180	220	260	300	340	380	420
P/a	12,800	7500	5000	3800	2800	2100	1700	1300

39. Heat conduction of asbestos; θ is the temperature in degrees Fahrenheit, C is the coefficient of conductivity.

θ	32	212	392	572	752	1112
C	1.048	1.346	1.451	1.499	1.548	1.644

40. Rubber-covered wires exposed to high external temperatures; C is the maximum current in amperes, A is the area of cross-section in sq. in.

C	3.2	5.9	9.0	22.0	42.0	68.0	84.0	102.0
A	0.001810	0.004072	0.007052	0.02227	0.05000	0.09442	0.1250	0.1595

41. Pressure-volume relation for an air compressor; p is the pressure, v is the volume.

p	18	21	26.5	33.5	44	62
v	0.635	0.556	0.475	0.397	0.321	0.243

42. Power delivered by an electric station; w is the average weight of coal consumed per hour per kilowatt delivered, f is the load factor.

f	0.25	0.20	0.15	0.10	0.05
w	2.843	3.012	3.293	3.856	5.545

43. Temperature at different depths in an artesian well; θ is the temperature in degrees C., d is the depth.

d	28	66	173	248	298	400	505	548
θ	11.71	12.90	16.40	20.00	22.20	23.75	26.45	27.70

44. Resistance of copper wire; R is the resistance in ohms per 1000 ft., D is the diameter of wire in mils.

D	289	182	102	57	32	18	10
R	0.126	0.317	1.010	3.234	10.26	32.80	105.1

45. Hysteresis losses in soft sheet iron subjected to an alternating magnetic flux; B is the flux density in kilolines per sq. in., P is the number of watts lost per cu. in. for 1 cycle per sec.

B	20	40	60	80	100	120
P	0.0022	0.0067	0.0128	0.0202	0.0289	0.0387

46. Volt-ampere characteristic of a 60 watt tungsten lamp: V is the number of volts, I is the number of milli-amperes.

V	2	5	10	20	30	40	50	60	70	80	90	100
I	49	80	117	180	227	272	311	348	383	414	443	473

V	110	120	130	140	150	160	170	180	190	200	210	220
I	501	526	553	577	597	618	639	663	682	702	722	743

47. Calibration of base metal pyrometer (40% Ni and 60% Cu); V is the number of millivolts, t is the temperature in degrees F.

V	0	2	4	6	8	10	12	14	16
t	0	146	255	320	396	475	553	634	714

48. Tests on drying of twine; t is the drying time in minutes (time of contact of twine with hot drum), W is the percentage of total water on bone dry twine at any time, E is the percentage of total water on bone dry twine at equilibrium, d is the diameter of the twine in ins.

(*a*) $d = 0.102$ ins., $E = 18.7\%$.

t	0	0.44	0.88	1.31	1.75
$W-E$	29.5	15.4	9.4	5.1	3.1

(*b*) $d = 0.158$, $E = 6.2\%$.

t	0	1.11	2.23	3.34	4.45	5.56
$W-E$	30.3	17.4	12.4	8.2	4.9	3.3

CHAPTER VII.

EMPIRICAL FORMULAS — PERIODIC CURVES.

85. Representation of periodic phenomena. — Periodic phenomena, such as alternating electric currents and alternating voltages, valve-gear motions, propagation of sound waves, heat waves, tidal observations, etc., may be represented graphically by curves composed of a repetition of congruent parts at certain intervals. Such a periodic curve may in turn be represented analytically by a periodic function of a variable, *i.e.*, by a function such that $f(x + k) = f(x)$, where k is the period. Thus the functions $\sin x$ and $\cos x$ have a period 2π, since $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$. Again, the function $\sin 5x$ has a period $2\pi/5$, since $\sin 5(x + 2\pi/5) = \sin(5x + 2\pi) = \sin 5x$, but the function $\sin x + \sin 5x$ has a period 2π, since $\sin(x + 2\pi) + \sin 5(x + 2\pi) = \sin x + \sin 5x$.

Now, any single-valued periodic function can, in general, be expressed by an infinite trigonometric series or Fourier's series of the form

$$\begin{aligned} y = f(x) = a_0 &+ a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots \\ &+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots, \end{aligned}$$

where the coefficients a_k and b_k may be determined if the function is known. This series has a period 2π. But usually the function is unknown. Thus, in the problems mentioned above, the curve may either be drawn by an oscillograph or by other instruments, or the values of the ordinates may be given by means of which the curve may be drawn. Our problem then is to represent this curve approximately by a series of the above form, containing a finite number of terms, and to find the approximate values of the coefficients a_k and b_k. The following sections will give some of the methods employed to determine these coefficients.

86. The fundamental and the harmonics of a trigonometric series. — In Fig. 86*a* we have drawn the curves $y = a_1 \cos x$, $y = b_1 \sin x$, and $y = a_1 \cos x + b_1 \sin x$.

The maximum height or amplitude of $y = a_1 \cos x$ is a_1 and the period is 2π. The amplitude of $y = b_1 \sin x$ is b_1 and the period is 2π. Now we may write

$$y = a_1 \cos x + b_1 \sin x = \sqrt{a_1^2 + b_1^2}\left[\frac{b_1}{\sqrt{a_1^2 + b_1^2}} \sin x + \frac{a_1}{\sqrt{a_1^2 + b_1^2}} \cos x\right],$$

and letting $\sqrt{a_1^2 + b_1^2} = c_1$, $\dfrac{b_1}{\sqrt{a_1^2 + b_1^2}} = \cos\phi_1$, $\dfrac{a_1}{\sqrt{a_1^2 + b_1^2}} = \sin\phi_1$,

we may write

$$y = c_1 \sin(x + \phi_1), \quad \text{where} \quad c_1 = \sqrt{a_1^2 + b_1^2}, \quad \phi_1 = \tan^{-1}\frac{a_1}{b_1}.$$

Here c_1 is the amplitude and ϕ_1 is called the phase. The wave represented by $y = c_1 \sin(x + \phi_1)$ is called the fundamental wave and $y = a_1 \cos x$, $y = b_1 \sin x$ are called its components.

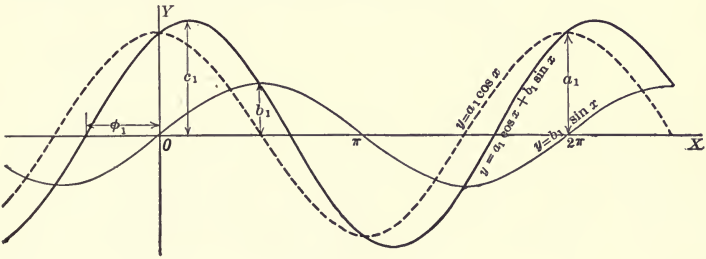


FIG. 86*a*.

Similarly, we may represent $y = a_k \cos kx$, $y = b_k \sin kx$,

and $$y = a_k \cos kx + b_k \sin kx = c_k \sin(kx + \phi_k),$$

where $$c_k = \sqrt{a_k^2 + b_k^2} \text{ and } \phi_k = \tan^{-1} a_k/b_k.$$

The wave represented by $y = c_k \sin(kx + \phi_k)$ is called the kth harmonic, its amplitude is c_k, its phase is ϕ_k, its period is $2\pi/k$, since

$$\sin\left[k\left(x + \frac{2\pi}{k}\right) + \phi_k\right] = \sin[kx + 2\pi + \phi_k] = \sin(kx + \phi_k),$$

and its frequency, or the number of complete waves in the interval 2π, is k.

The trigonometric series is often written in the form

$$y = c_0 + c_1 \sin(x + \phi_1) + c_2 \sin(2x + \phi_2) + \cdots + c_n \sin(nx + \phi_n) + \cdots,$$

showing explicitly the expressions for the fundamental wave and the successive harmonics. The more complex wave represented by this expression may be built up by a combination of the waves represented by the various harmonics. Fig. 86*b* shows how the wave for the equation

$$y = 2\sin\left(x + \frac{\pi}{6}\right) + \sin\left(2x - \frac{2\pi}{3}\right) + \tfrac{1}{2}\sin\left(3x + \frac{3\pi}{4}\right),$$

or

$$y = \cos x - \frac{\sqrt{3}}{2}\cos 2x + \frac{\sqrt{2}}{4}\cos 3x + \sqrt{3}\sin x - \tfrac{1}{2}\sin 2x - \frac{\sqrt{2}}{4}\sin 3x$$

is built up as the combination of the fundamental and the second and third harmonics, and how the fundamental wave is modified by the addition of the harmonic waves.

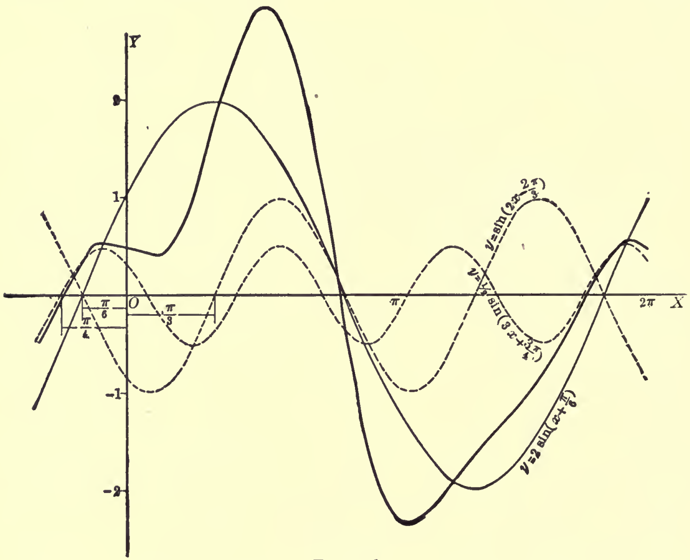


FIG. 86*b*.

In the case of alternating currents or voltages, the portion of the wave extending from $x = \pi$ to $x = 2\pi$ is merely a repetition below the x-axis of the portion of the wave extending from $x = 0$ to $x = \pi$; this is illustrated in Fig. 86*c* where the values of the ordinate at $x = x_r + \pi$ is minus the value of the ordinate at $x = x_r$. Since

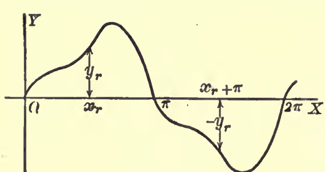


FIG. 86*c*.

$$\sin(k[x + \pi] + \phi_k) = \sin(kx + \phi_k + k\pi)$$
$$= +\sin(kx + \phi_k) \text{ if } k \text{ is even}$$
$$= -\sin(kx + \phi_k) \text{ if } k \text{ is odd,}$$

the series can contain only the odd harmonics and has the form

$$y = c_0 + c_1 \sin(x + \phi_1) + c_3 \sin(3x + \phi_3) + c_5 \sin(5x + \phi_5) + \cdots,$$

or

$$y = a_0 + a_1 \cos x + a_3 \cos 3x + a_5 \cos 5x + \cdots$$
$$+ b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \cdots.$$

87. Determination of the constants when the function is known.— If, in the series

$$y = f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots,$$

we multiply both sides by dx and integrate between the limits 0 and 2π, we have

$$\int_0^{2\pi} y\,dx = a_0 \int_0^{2\pi} dx + a_1 \int_0^{2\pi} \cos x\,dx + \ldots + a_n \int_0^{2\pi} \cos nx\,dx + \cdots$$
$$+ b_1 \int_0^{2\pi} \sin x\,dx + \ldots + b_n \int_0^{2\pi} \sin nx\,dx + \cdots$$
$$= a_0 \Big| x \Big|_0^{2\pi} + a_1 \Big| \sin x \Big|_0^{2\pi} + \cdots + \frac{a_n}{n} \Big| \sin nx \Big|_0^{2\pi} + \cdots$$
$$- b_1 \Big| \cos x \Big|_0^{2\pi} - \cdots - \frac{b_n}{n} \Big| \cos nx \Big|_0^{2\pi} - \cdots$$

$= 2\pi a_0$, since all the other terms vanish.

If we multiply both sides by $\cos kx\,dx$ and integrate between the limits 0 and 2π, we have

$$\int_0^{2\pi} y \cos kx\,dx = a_0 \int_0^{2\pi} \cos kx\,dx + \cdots + a_k \int_0^{2\pi} \cos^2 kx\,dx + \cdots$$
$$+ a_n \int_0^{2\pi} \cos nx \cos kx\,dx + \cdots \quad + b_n \int_0^{2\pi} \sin nx \cos kx\,dx + \cdots$$
$$= \frac{a_0}{k} \Big| \sin kx \Big|_0^{2\pi} + \cdots + \frac{a_k}{2} \Big| x + \frac{\sin 2kx}{2k} \Big|_0^{2\pi} + \cdots$$
$$+ \frac{a_n}{2} \Big| \frac{\sin (n-k)x}{n-k} + \frac{\sin (n+k)x}{n+k} \Big|_0^{2\pi} + \cdots$$
$$- \frac{b_n}{2} \Big| \frac{\cos (n-k)x}{n-k} + \frac{\cos (n+k)x}{n+k} \Big|_0^{2\pi} - \cdots$$

$= \pi a_k$, since all the other terms vanish.

Similarly, if we multiply both sides by $\sin kx\,dx$ and integrate between the limits 0 and 2π, we have

$$\int_0^{2\pi} y \sin kx\,dx = a_0 \int_0^{2\pi} \sin kx\,dx + \cdots + a_n \int_0^{2\pi} \cos nx \sin kx\,dx + \cdots$$
$$+ \cdots + b_k \int_0^{2\pi} \sin^2 kx\,dx + \cdots + b_n \int_0^{2\pi} \sin nx \sin kx\,dx + \cdots$$
$$= -\frac{a_0}{k} \Big| \cos kx \Big|_0^{2\pi} - \cdots + \frac{b_k}{2} \Big| x - \frac{\sin 2kx}{2k} \Big|_0^{2\pi} + \cdots$$
$$- \frac{a_n}{2} \Big| \frac{\cos (k-n)x}{k-n} + \frac{\cos (k+n)x}{k+n} \Big|_0^{2\pi} - \cdots$$
$$+ \frac{b_n}{2} \Big| \frac{\sin (n-k)x}{n-k} - \frac{\sin (n+k)x}{n+k} \Big|_0^{2\pi} + \cdots$$

$= \pi b_k$, since all the other terms vanish.

Collecting our results, we have

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} y\,dx, \quad a_k = \frac{1}{\pi}\int_0^{2\pi} y\cos kx\,dx, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} y\sin kx\,dx,$$

where $k = 1, 2, 3, \ldots$. Each coefficient may thus be independently determined and thus each individual harmonic can be calculated without calculating the preceding harmonics.

88. Determination of the constants when the function is unknown. — In our problems the function is unknown, and the periodic curve is drawn mechanically or a set of ordinates are given by means of which the curve may be approximately drawn. We shall represent the curve by a trigonometric series with a finite number of terms. We divide the interval from $x = 0$ to $x = 2\pi$ into n equal intervals and measure the first n ordinates; these are represented by the table

x	0 x_0	$\frac{2\pi}{n}$ x_1	$\frac{4\pi}{n}$ x_2	$\frac{6\pi}{n}$ x_3	$\ldots$	$r\frac{2\pi}{n}$ x_r	$\ldots$	$(n-1)\frac{2\pi}{n}$ x_{n-1}
y	y_0	y_1	y_2	y_3	$\ldots$	y_r	$\ldots$	y_{n-1}

We wish to determine the constants in the equation

$$\begin{aligned} y = a_0 + a_1\cos x + \cdots + a_k\cos kx + \cdots \\ + b_1\sin x + \cdots + b_k\sin kx + \cdots, \end{aligned}$$

where the number of terms is n, so that the corresponding curve will pass through the n points given in the table. Substituting the n sets of values of x and y in this equation, we get n linear equations in the a's and b's of the form

$$\begin{aligned} y_r = a_0 + a_1\cos x_r + \cdots + a_k\cos kx_r + \cdots \\ + b_1\sin x_r + \cdots + b_k\sin kx_r + \cdots, \end{aligned}$$

where r takes in succession the values $0, 1, 2, \ldots, n-1$. We may now solve these n equations for the a's and b's.

We shall first state two theorems in Trigonometry concerning the sum of the cosines or sines of n angles which are in arithmetic progression, viz.:

$$\begin{aligned} \sum\cos(\alpha + r\beta) &= \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + \cdots \\ &+ \cos(\alpha + [n-1]\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \frac{n-1}{2}\beta\right), \end{aligned}$$

$$\sum \sin(\alpha + r\beta) = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha + 2\beta) + \cdots$$

$$+ \sin(\alpha + [n-1]\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \sin\left(\alpha + \frac{n-1}{2}\beta\right).*$$

If we let $\alpha = 0$ and $\beta = l\frac{2\pi}{n}$, these become

$$\sum \cos rl\frac{2\pi}{n} = \frac{\sin l\pi}{\sin\frac{l\pi}{n}} \cos\frac{l(n-1)\pi}{n} = 0, \text{ since } \sin l\pi = 0,$$

$$\sum \sin rl\frac{2\pi}{n} = \frac{\sin l\pi}{\sin\frac{l\pi}{n}} \sin\frac{l(n-1)\pi}{n} = 0, \text{ since } \sin l\pi = 0,$$

* We may prove these theorems as follows:

By means of the well-known trigonometric identities

$2\cos u \sin v = \sin(u+v) - \sin(u-v)$, $\quad 2\sin u\sin v = \cos(u-v) - \cos(u+v)$

we may write the identities

$$2\cos\alpha\sin\frac{\beta}{2} = \sin\left(\alpha+\frac{\beta}{2}\right) - \sin\left(\alpha-\frac{\beta}{2}\right).$$

$$2\cos(\alpha+\beta)\sin\frac{\beta}{2} = \sin\left(\alpha+\frac{3\beta}{2}\right) - \sin\left(\alpha+\frac{\beta}{2}\right).$$

$$2\cos(\alpha+2\beta)\sin\frac{\beta}{2} = \sin\left(\alpha+\frac{5\beta}{2}\right) - \sin\left(\alpha+\frac{3\beta}{2}\right).$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$2\cos(\alpha+[n-1]\beta)\sin\frac{\beta}{2} = \sin\left(\alpha+\frac{2n-1}{2}\beta\right) - \sin\left(\alpha+\frac{2n-3}{2}\beta\right).$$

Adding, we get

$$2\sin\frac{\beta}{2}\sum\cos(\alpha+r\beta) = \sin\left(\alpha+\frac{2n-1}{2}\beta\right) - \sin\left(\alpha-\frac{\beta}{2}\right)$$

$$= 2\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n}{2}\beta.$$

$$\therefore \sum\cos(\alpha+r\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha+\frac{n-1}{2}\beta\right).$$

$$2\sin\alpha\sin\frac{\beta}{2} = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{\beta}{2}\right).$$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{3\beta}{2}\right).$$

$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{3\beta}{2}\right) - \cos\left(\alpha+\frac{5\beta}{2}\right).$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$2\sin(\alpha+[n-1]\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{2n-3}{2}\beta\right) - \cos\left(\alpha+\frac{2n-1}{2}\beta\right).$$

Adding, we get

$$2\sin\frac{\beta}{2}\sum\sin(\alpha+r\beta) = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{2n-1}{2}\beta\right)$$

$$= 2\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}.$$

$$\therefore \sum\sin(\alpha+r\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha+\frac{n-1}{2}\beta\right).$$

for all values of l except

$$l = 0, \text{ when } \sum \cos rl\frac{2\pi}{n} = \sum \cos 0 = n,$$

$$l = n, \text{ when } \sum \cos rl\frac{2\pi}{n} = \sum \cos 2r\pi = n.$$

Since $x_r = r\frac{2\pi}{n}$, we may finally state that

$$\sum \cos lx_r = 0, \text{ except when } l = 0 \text{ or } l = n$$

$$= n, \text{ when } l = 0 \text{ or } l = n.$$

$$\sum \sin lx_r = 0 \text{ for all values of } l.$$

To determine a_0 we merely add the n equations, and get

$$\sum y_r = na_0 + \cdots + a_k \sum \cos kx_r + \cdots + a_k \sum \sin kx_r + \cdots$$

$$= na_0, \text{ since all the other terms vanish.}$$

To determine a_k we multiply each of the n equations by the coefficient of a_k in that equation, *i.e.*, by $\cos kx_r$, and add the n resulting equations; we get

$$\sum y_r \cos kx_r = a_0 \sum \cos kx_r + \cdots + a_k \sum \cos^2 kx_r + \cdots$$

$$+ a_p \sum \cos px_r \cos kx_r + \cdots + b_p \sum \sin px_r \cos kx_r + \cdots.$$

Now,

$$\sum \cos kx_r = 0;$$

$$\sum \cos px_r \cos kx_r{}^* = \tfrac{1}{2}\sum \cos (p+k)\,x_r + \tfrac{1}{2}\sum \cos (p-k)x_r = 0;$$

$$\sum \sin px_r \cos kx_r{}^* = \tfrac{1}{2}\sum \sin (p+k)\,x_r + \tfrac{1}{2}\sum \sin (p-k)\,x_r = 0;$$

$$\sum \cos^2 kx_r = \sum \frac{1}{2}(1 + \cos 2\,kx_r) = \frac{n}{2} + \frac{1}{2}\sum \cos 2\,kx_r = \frac{n}{2}, \text{ if } k \neq \frac{n}{2}$$

$$= n, \text{ if } k = \frac{n}{2}.$$

* We use the trigonometric identities

$$2 \cos u \cos v = \cos (u+v) + \cos (u-v).$$
$$2 \sin u \cos v = \sin (u+v) + \sin (u-v).$$
$$2 \sin u \sin v = \cos (u-v) - \cos (u+v).$$

Hence, $$\sum y_r \cos kx_r = \frac{n}{2} a_k, \text{ except when } k = \frac{n}{2}$$
$$= na_k, \text{ when } k = \frac{n}{2}.$$

To determine b_k we multiply each of the n equations by the coefficient of b_k in that equation, *i.e.*, by $\sin kx_r$, and add the n resulting equations: we get

$$\sum y_r \sin kx_r = a_0 \sum \sin kx_r + \cdots a_p \sum \cos px_r \sin kx_r + \cdots$$
$$+ b_k \sum \sin^2 kx_r + \cdots + b_p \sum \sin px_r \sin kx_r + \cdots.$$

Now,

$$\sum \sin kx_r = 0;$$

$$\sum \cos px_r \sin kx_r{}^* = \tfrac{1}{2} \sum \sin (k + p) x_r + \tfrac{1}{2} \sum \sin (k - p) x_r = 0;$$

$$\sum \sin px_r \sin kx_r{}^* = \tfrac{1}{2} \sum \cos (p - k) x_r - \tfrac{1}{2} \sum \cos (p + k) x_r = 0;$$

$$\sum \sin^2 kx_r = \sum \frac{1}{2} (1 - \cos 2 kx_r) = \frac{n}{2} - \frac{1}{2} \sum \cos 2 kx_r = \frac{n}{2}, \text{ if } k \neq \frac{n}{2}$$
$$= 0, \text{ if } k = \frac{n}{2}.$$

Hence, $$\sum y_r \sin kx_r = \frac{n}{2} b_k.$$

Collecting our results, we have finally

$$a_0 = \frac{1}{n} \sum y_r = \frac{1}{n} (y_0 + y_1 + y_2 + \cdots + y_{n-1}),$$

$$a_{\frac{n}{2}} = \frac{1}{n} \sum y_r \cos \frac{n}{2} x_r = \frac{1}{n} \sum y_r \cos r\pi = \frac{1}{n} (y_0 - y_1 + y_2 - y_3 + \cdots - y_{n-1}),$$

$$a_k = \frac{2}{n} \sum y_r \cos kx_r = \frac{2}{n} (y_0 \cos kx_0 + y_1 \cos kx_1 + \cdots + y_{n-1} \cos kx_{n-1}),$$

$$b_k = \frac{2}{n} \sum y_r \sin kx_r = \frac{2}{n} (y_0 \sin kx_0 + y_1 \sin kx_1 + \cdots + y_{n-1} \sin kx_{n-1}).$$

* We use the trigonometric identities

$$2 \cos u \cos v = \cos (u + v) + \cos (u - v).$$
$$2 \sin u \cos v = \sin (u + v) + \sin (u - v).$$
$$2 \sin u \sin v = \cos (u - v) - \cos (u + v).$$

If n is an *even* integer, our periodic curve is now represented by the equation

$$y = a_0 + a_1 \cos x + \cdots + a_k \cos kx + \cdots + a_{\frac{n}{2}} \cos \frac{n}{2} x$$
$$+ b_1 \sin x + \cdots + b_k \sin kx + \cdots + b_{\frac{n}{2}-1} \sin\left(\frac{n}{2} - 1\right) x.$$

The n coefficients are determined as above. Thus —

a_0 is the average value of the n ordinates.

$a_{\frac{n}{2}}$ is the average value of the n ordinates taken alternately plus and minus.

a_k or b_k is twice the average value of the products formed by multiplying each ordinate by the *cosine* or *sine* of k times the corresponding value of x.*

We note that each coefficient is determined independently of all the others.

If we wished to represent the periodic curve by a Fourier's series containing n terms, but had measured m ordinates, where $m > n$, we should have to determine the coefficients by the method of least squares. The values of the ordinates as computed from this series will agree as closely as possible with the values of the measured ordinates. It may be shown that the expressions for the coefficients obtained by the method of least squares have the same form as those derived above.†

* We may also derive the values of the coefficients as follows: In Art. **87**, we have shown that

$$\int_0^{2\pi} y \cos kx \, dx = a_k \int_0^{2\pi} \cos^2 kx \, dx,$$

since all the other terms vanish.

If we replace the integrals by sums, and take for dx the interval $2\pi/n$, this becomes

$$\sum y_r \cos kx_r = a_k \sum \cos^2 kx_r = \frac{n}{2} a_k, \text{ if } k \neq 0 \text{ or } k \neq \frac{n}{2}$$
$$= n a_k, \text{ if } k = 0 \text{ or } k = \frac{n}{2}.$$

Hence, $\sum y_r = n a_0, \quad \sum y_r \cos \frac{n}{2} x_r = n a_{\frac{n}{2}}, \quad \sum y_r \cos kx_r = \frac{n}{2} a_k.$

Similarly we may show that $\sum y_r \sin kx_r = \frac{n}{2} b_k.$

† See A Course in Fourier's Analysis and Periodogram Analysis by G. A. Carse and G. Shearer.

We shall illustrate the use of the above formulas for the coefficients by finding the fifth harmonic in the equation of the periodic curve passing through the 12 points given by the following data (Fig. 89).

x	y	$\cos 5x$	$\sin 5x$	$y \cos 5x$	$y \sin 5x$
0°	9.3	1.000	0.000	9.30	0.00
30°	15.0	−0.866	0.500	−12.99	7.50
60°	17.4	0.500	−0.866	8.70	−15.07
90°	23.0	0.000	1.000	0.00	23.00
120°	37.0	−0.500	−0.866	−18.50	−32.04
150°	31.0	0.866	0.500	26.85	15.50
180°	15.3	−1.000	0.000	−15.30	0.00
210°	4.0	0.866	−0.500	3.46	− 2.00
240°	− 8.0	−0.500	0.866	4.00	− 6.93
270°	−13.2	0.000	−1.000	0.00	13.20
300°	−14.2	0.500	0.866	− 7.10	−12.30
330°	− 6.0	−0.866	−0.500	5.20	3.00
			$\Sigma =$	3.62	− 6.14

$$a_5 = \tfrac{2}{12} \sum y_r \cos 5\, x_r = 0.60; \qquad b_5 = \tfrac{2}{12} \sum y_r \sin 5\, x_r = -1.02.$$

Hence the fifth harmonic is $0.60 \cos 5\,x - 1.02 \sin 5\,x$.

It is evident that the labor involved in the direct determination of the coefficients by the above formulas is very great. This labor may be reduced to a minimum by arranging the work in tabular form. These forms follow the methods devised by Runge * for periodic curves involving both even and odd harmonics (Art. 89), and by S. P. Thompson † for periodic curves involving only odd harmonics (Art. 90).

89. Numerical evaluation of the coefficients. Even and odd harmonics. —

(I) *Six-ordinate scheme.* — Given the curve and wishing to determine the first three harmonics, *i.e.*, the 6 coefficients in the equation

$$y = a_0 + a_1 \cos x + a_2 \cos 2\,x + a_3 \cos 3\,x + b_1 \sin x + b_2 \sin 2\,x,$$

we divide the period from $x = 0°$ to $x = 360°$ ‡ into 6 equal parts and

* Zeit. f. Math. u. Phys., xlviii. 443 (1903), lii. 117 (1905); Erläuterung des Rechnungsformulars, u.s.w., Braunschweig, 1913.

† Proc. Phys. Soc., xix. 443, 1905; The Electrician, 5th May, 1905.

‡ If the period is taken equal to $2\,\pi/m$ instead of $2\,\pi$, the representing trigonometric series has the form

$$y = a_0 + a_1 \cos m\theta + a_2 \cos 2\,m\theta + \cdots$$
$$+ b_1 \sin m\theta + b_2 \sin 2\,m\theta + \cdots,$$

where θ represents abscissas. By the substitution $m\theta = x$ or $\theta = x/m$, the series becomes

$$y = a_0 + a_1 \cos x + a_2 \cos 2\,x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2\,x + \cdots,$$

and this has a period $2\,\pi$. The abscissas from $\theta = 0$ to $\theta = 2\,\pi/m$ now become the abscissas from $x = 0$ to $x = 2\,\pi$, and we proceed to determine the coefficients in the second series as outlined. Having determined the coefficients, we finally replace x by $m\theta$.

measure the ordinates at the beginning of each interval; let these be represented by the following table:

x	$0°$	$60°$	$120°$	$180°$	$240°$	$300°$
y	y_0	y_1	y_2	y_3	y_4	y_5

Here $n = 6$, and using the formulas on p. 177, we have

$$
\begin{aligned}
6a_0 &= y_0 + y_1 + y_2 + y_3 + y_4 + y_5 \\
6a_3 &= y_0 - y_1 + y_2 - y_3 + y_4 - y_5 \\
3a_1 &= y_0\cos 0° + y_1\cos 60° + y_2\cos 120° + y_3\cos 180° + y_4\cos 240° + y_5\cos 300° \\
3a_2 &= y_0\cos 0° + y_1\cos 120° + y_2\cos 240° + y_3\cos 360° + y_4\cos 480° + y_5\cos 600° \\
3b_1 &= y_0\sin 0° + y_1\sin 60° + y_2\sin 120° + y_3\sin 180° + y_4\sin 240° + y_5\sin 300° \\
3b_2 &= y_0\sin 0° + y_1\sin 120° + y_2\sin 240° + y_3\sin 360° + y_4\sin 480° + y_5\sin 600°
\end{aligned}
$$

We arrange the y's in two rows,

	y_0	y_1	y_2	y_3
		y_5	y_4	
Sum	v_0	v_1	v_2	v_3
Diff.		w_1	w_2	

where the v's are the sums and the w's are the differences of the quantities standing in the same vertical column; thus, $v_0 = y_0$, $v_1 = y_1 + y_5$, $w_1 = y_1 - y_5$, etc. Since $\cos 240° = \cos 120°$, $\cos 300° = \cos 60°$, etc. We may now write

$$
\begin{aligned}
6a_0 &= v_0 + v_1 + v_2 + v_3 \\
6a_3 &= v_0 - v_1 + v_2 - v_3 \\
3a_1 &= v_0 + v_1\cos 60° + v_2\cos 120° + v_3\cos 180° \\
3a_2 &= v_0 + v_1\cos 120° + v_2\cos 240° + v_3\cos 360° \\
3b_1 &= w_1\sin 60° + w_2\sin 120° \\
3b_2 &= w_1\sin 120° + w_2\sin 240°
\end{aligned}
$$

We arrange the v's and w's in rows,

	v_0	v_1	w_1
	v_3	v_2	w_2
Sum	p_0	p_1	r_1
Diff.	q_0	q_1	s_1

and we now write

$$6a_0 = p_0 + p_1, \qquad 6a_3 = q_0 - q_1,$$

$$3a_1 = q_0 + \tfrac{1}{2}q_1, \qquad 3a_2 = p_0 - \tfrac{1}{2}p_1,$$

$$3b_1 = \frac{\sqrt{3}}{2}r_1, \qquad 3b_2 = \frac{\sqrt{3}}{2}s_1.$$

Example. Determine the first three harmonics for the following data taken from Fig. 86*b*.

x	0°	60°	120°	180°	240°	300°
y	0.47	1.77	2.20	−2.20	−1.64	−0.49

	0.47	1.77	2.20	−2.20
		−0.49	−1.64	
v	0.47	1.28	0.56	−2.20
w		2.26	3.84	

	0.47	1.28		2.26
	−2.20	0.56		3.84
p	−1.73	1.84	r	6.10
q	2.67	0.72	s	−1.58

$$6\,a_0 = 0.11, \qquad 6\,a_3 = 1.95, \qquad 3\,a_1 = 3.03,$$
$$3\,a_2 = -2.65, \qquad 3\,b_1 = 5.28, \qquad 3\,b_2 = -1.37.$$

Hence, $a_0 = 0.02,\quad a_1 = 1.01,\quad a_2 = -0.88,\quad a_3 = 0.33,$
$b_1 = 1.76,\quad b_2 = -0.46,$

and
$$y = 0.02 + 1.01\cos x - 0.88\cos 2\,x + 0.33\cos 3\,x + 1.76\sin x - 0.46\sin 2\,x.$$

The equation from which the curve in Fig. 86*b* was plotted was

$$y = 2\sin\left(x + \frac{\pi}{6}\right) + \sin\left(2\,x - \frac{2\,\pi}{3}\right) + \frac{1}{2}\sin\left(3\,x + \frac{3\,\pi}{4}\right)$$
$$= \cos x - 0.87\cos 2\,x + 0.35\cos 3\,x + 1.73\sin x + 0.50\sin 2\,x - 0.35\sin 3\,x.$$

We observe the close agreement between the two sets of coefficients, the small discrepancies being due to the approximate measurements of the ordinates for our example.

(II) *Twelve-ordinate scheme.*—Given the curve and wishing to determine the first six harmonics, *i.e.*, the 12 coefficients in the equation

$$y = a_0 + a_1\cos x + a_2\cos 2\,x + a_3\cos 3\,x + a_4\cos 4\,x + a_5\cos 5\,x + a_6\cos 6\,x + b_1\sin x + b_2\sin 2\,x + b_3\sin 3\,x + b_4\sin 4\,x + b_5\sin 5\,x,$$

we divide the interval from $x = 0$ to $x = 360°$ into 12 equal parts and measure the ordinates at the beginning of each interval; let these be represented by the following table:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
y	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}

Here $n = 12$, and the formulas for the coefficients give

$$12\,a_0 = y_0 + y_1 + y_2 + \cdots + y_{11}$$
$$12\,a_6 = y_0 - y_1 + y_2 - \cdots - y_{11}$$
$$6\,a_1 = y_0\cos 0° + y_1\cos 30° + y_2\cos 60° + \cdots + y_{11}\cos 330°$$
$$6\,a_2 = y_0\cos 0° + y_1\cos 60° + y_2\cos 120° + \cdots + y_{11}\cos 660°$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$6\,b_1 = y_0\sin 0° + y_1\sin 30° + y_2\sin 60° + \cdots + y_{11}\sin 330°$$
$$6\,b_2 = y_0\sin 0° + y_1\sin 60° + y_2\sin 120° + \cdots + y_{11}\sin 660°$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

If we arrange the y's in two rows

	y_0	y_1	y_2	y_3	y_4	y_5	y_6
		y_{11}	y_{10}	y_9	y_8	y_7	
Sum	v_0	v_1	v_2	v_3	v_4	v_5	v_6
Diff.		w_1	w_2	w_3	w_4	w_5	

and remember that $\cos 330^\circ = \cos 30^\circ$, $\sin 330^\circ = -\sin 30^\circ$, etc., the above equations may be written

$$\begin{aligned}
12\,a_0 &= v_0 + v_1 \qquad + v_2 \qquad + \cdots + v_6\\
12\,a_6 &= v_0 - v_1 \qquad + v_2 \qquad - \cdots + v_6\\
6\,a_1 &= v_0 + v_1 \cos 30^\circ + v_2 \cos 60^\circ + \cdots + v_6 \cos 180^\circ\\
6\,a_2 &= v_0 + v_1 \cos 60^\circ + v_2 \cos 120^\circ + \cdots + v_6 \cos 360^\circ\\
&\cdots\cdots\cdots\cdots\cdots\\
6\,b_1 &= w_1 \sin 30^\circ + w_2 \sin 60^\circ + \cdots + w_5 \sin 150^\circ\\
6\,b_2 &= w_1 \sin 60^\circ + w_2 \sin 120^\circ + \cdots + w_5 \sin 300^\circ\\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

If we now arrange the v's and w's in two rows

	v_0	v_1	v_2	v_3	w_1	w_2	w_3
	v_6	v_5	v_4		w_5	w_4	
Sum	p_0	p_1	p_2	p_3	r_1	r_2	r_3
Diff.	q_0	q_1	q_2		s_1	s_2	

the equations may be written

$$\begin{aligned}
12\,a_0 &= q_0 + q_1 \qquad + q_2 \qquad + q_3\\
12\,a_6 &= p_0 - p_1 \qquad + p_2 \qquad - p_3\\
6\,a_1 &= q_0 + q_1 \cos 30^\circ + q_2 \cos 60^\circ\\
6\,a_2 &= p_0 + p_1 \cos 60^\circ + p_2 \cos 120^\circ + p_3 \cos 180^\circ\\
&\cdots\cdots\cdots\cdots\cdots\\
6\,b_1 &= r_1 \sin 30^\circ \qquad + r_2 \sin 60^\circ \qquad + r_3 \sin 90^\circ\\
6\,b_2 &= s_1 \sin 60^\circ \qquad + s_2 \sin 120^\circ\\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Finally, if we arrange the p's, q's, and r's as follows:

	p_0	p_1		r_1	q_0
	p_2	p_3		r_3	q_2
Sum	l_0	l_1	Diff.	t_1	t_2

the equations become

$12\,a_0 = l_0 + l_1.$

$6\,a_1 = q_0 + q_1 \sin 60^\circ + q_2 \sin 30^\circ.$

$6\,a_2 = (p_0 - p_3) + (p_1 - p_2) \sin 30^\circ.$

$6\,a_3 = t_2.$

$6\,b_1 = r_1 \sin 30^\circ + r_2 \sin 60^\circ + r_3.$

$6\,b_2 = (s_1 + s_2) \sin 60^\circ.$

$12\,a_6 = l_0 - l_1.$

$6\,a_5 = q_0 - q_1 \sin 60^\circ + q_2 \sin 30^\circ.$

$6\,a_4 = (p_0 + p_3) - (p_1 + p_2) \sin 30^\circ.$

$6\,b_3 = t_1.$

$6\,b_5 = r_1 \sin 30^\circ - r_2 \sin 60^\circ + r_3.$

$6\,b_{.} = (s_1 - s_2) \sin 60^\circ.$

We may now arrange the above scheme in a computing form as follows:

Ordinates	y_0	y_1	y_2	y_3	y_4	y_5	y_6
		y_{11}	y_{10}	y_9	y_8	y_7	
Sum	v_0	v_1	v_2	v_3	v_4	v_5	v_6
Diff.		w_1	w_2	w_3	w_4	w_5	

	v_0	v_1	v_2	v_3	w_1	w_2	w_3
	v_6	v_5	v_4		w_5	w_4	
Sum	p_0	p_1	p_2	p_3	r_1	r_2	r_3
Diff.	q_0	q_1	q_2		s_1	s_2	

	p_0	p_1		r_1	q_0
	p_2	p_3		r_3	q_2
Sum	l_0	l_1	Diff.	t_1	t_2

Multipliers of the quantities in the same horizontal rows before these are entered	Cosine terms							Sine terms				
sin 30° = 0.5	q_2		$-p_2$	p_1				r_1				
sin 60° = 0.866		q_1							r_2	s_1	s_2	
sin 90° = 1.0	q_0		p_0	$-p_3$	t_2	l_0	l_1	r_3				t_1
Sum of 1st column	....		....			....		....		......		
Sum of 2d column	....		....			....		....		......		
Sum	$6a_1$		$6a_2$		$6a_3$	$12a_0$		$6b_1$		$6b_2$		$6b_3$
Difference	$6a_5$		$6a_4$			$12a_6$		$6b_5$		$6b_4$		

Checks: $y_0 = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6.$

$$y_1 - y_{11} = (b_1 + b_5) + \sqrt{3}\,(b_2 + b_4) + 2\,b_3.$$

Result: $y = a_0 + a_1 \cos x + a_2 \cos 2\,x + \cdots + a_6 \cos 6\,x$
$+ b_1 \sin x + b_2 \sin 2\,x + \cdots + b_5 \sin 5\,x.$

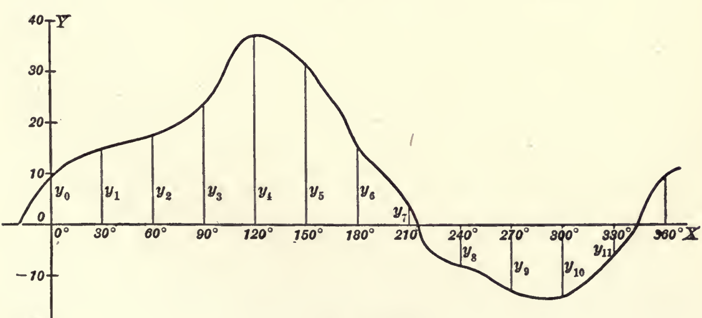


FIG. 89.

Example. In the periodic curve of Fig. 89, the interval from $x = 0°$ to $x = 360°$ is divided into 12 equal parts and the ordinates y_0 to y_{11} are measured.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
y	9.3	15.0	17.4	23.0	37.0	31.0	15.3	4.0	−8.0	−13.2	−14.2	−6.0

We shall determine the first six harmonics by the above scheme.

Ordinates	9.3	15.0	17.4	23.0	37.0	31.0	15.3
		−6.0	−14.2	−13.2	−8.0	4.0	
Sum (v)	9.3	9.0	3.2	9.8	29.0	35.0	15.3
Diff. (w)		21.0	31.6	36.2	45.0	27.0	

	9.3	9.0	3.2	9.8		21.0	31.6	36.2
	15.3	35.0	29.0			27.0	45.0	
Sum (p)	24.6	44.0	32.2	9.8	(r)	48.0	76.6	36.2
Diff. (q)	−6.0	−26.0	−25.8		(s)	−6.0	−13.4	

	24.6	44.0		48.1	−6.0
	32.2	9.8		36.2	−25.8
Sum (l)	56.8	53.8	Diff. (t)	11.9	19.8

Multipliers	Cosine terms				Sine terms		
0.5 0.866 1.0	−12.9 −22.5 −6.0	−16.1 22.0 24.6 −9.8	19.8	56.8 53.8	24.0 66.3 36.2	−5.2 −11.6	11.9
Sum of 1st col. Sum of 2d col.	−18.9 −22.5	8.5 12.2		56.8 53.8	60.2 66.3	− 5.2 −11.6	
Sum Diff.	$-41.4=6\,a_1$ $3.6=6\,a_5$	$20.7=6\,a_2$ $-3.7=6\,a_4$	19.8 $=6\,a_3$	$110.6=12\,a_0$ $3.0=12\,a_6$	$126.5=6\,b_1$ $-6.1=6\,b_5$	$-16.8=6\,b_2$ $6.4=6\,b_4$	11.9 $=6\,b_3$

$a_1 = -6.90$, $a_2 = 3.45$, $a_3 = 3.30$, $a_0 = 9.22$, $b_1 = 21.08$, $b_2 = -2.80$, $b_3 = 1.98$,
$a_5 = 0.60$, $a_4 = -0.62$, $a_6 = 0.25$, $b_5 = -1.02$, $b_4 = 1.07$.

Check: $9.3 = 9.22 - 6.90 + 3.45 + 3.30 - 0.62 + 0.60 + 0.25 = 9.30.$
$21.0 = (21.08 - 1.02) + 1.732\,(-2.80 + 1.07) + 2\,(1.98) = 21.02.$

Result: *

$$y = 9.22 - 6.90 \cos x + 3.45 \cos 2x + 3.30 \cos 3x - 0.62 \cos 4x + 0.60 \cos 5x + 0.25 \cos 6x + 21.08 \sin x - 2.80 \sin 2x + 1.98 \sin 3x + 1.07 \sin 4x - 1.02 \sin 5x,$$

or

$$y = 9.22 + 22.18 \sin (x - 18.12°) - 4.44 \sin (2x - 50.93°) + 3.85 \sin (3x + 59.04°) + 1.24 \sin (4x - 30.09°) - 1.18 \sin (5x - 30.47°) - 0.25 \sin (6x - 90°).$$

* The coefficients of the fifth harmonic agree with those found by the direct process in Art. 88. The time and labor spent in the computation of all six harmonics by means of the above computing form is much less than that spent in the determination of the fifth harmonic alone by the direct process in Art. 88.

The last result was obtained by using the relations

$$a_k \cos kx + b_k \sin kx = c_k \sin (kx + \phi_k),$$

where $$c_k = \sqrt{a_k^2 + b_k^2} \quad \text{and} \quad \phi_k = \tan^{-1} \frac{a_k}{b_k}.$$

(III) *Twenty-four-ordinate scheme.*—Given the curve and wishing to find the first 12 harmonics, *i.e.*, the 24 coefficients in the equation

$$\begin{aligned} y = a_0 + a_1 \cos x + a_2 \cos 2\,x + \cdots + a_{12} \cos 12\,x \\ + b_1 \sin x + b_2 \sin 2\,x + \cdots + b_{11} \sin 11\,x, \end{aligned}$$

we divide the interval from $x = 0°$ to $x = 360°$ into 24 equal parts and measure the ordinates at the beginning of each interval; let these be represented by the following table:

x	$0°$	$15°$	$30°$	$45°$	$\cdots$	$330°$	$345°$
y	y_0	y_1	y_2	y_3	$\cdots$	y_{22}	y_{23}

If we use the same method as that employed in deriving the 12-ordinate scheme, we shall arrive at the following 24-ordinate computing form. This form is self-explanatory.

$$\begin{array}{llllll}
\text{Ordinates} & y_0 & y_1 & y_2 & \cdots & y_{11} & y_{12} \\
 & & y_{23} & y_{22} & \cdots & y_{13} & \\ \hline
\text{Sum} & v_0 & v_1 & v_2 & \cdots & v_{11} & v_{12} \\
\text{Diff.} & & w_1 & w_2 & \cdots & w_{11} &
\end{array}$$

$$\begin{array}{lllllllllllll}
 & v_0 & v_1 & \cdots & v_5 & v_6 & \quad & w_1 & w_2 & \cdots & w_5 & w_6 \\
 & v_{12} & v_{11} & \cdots & v_7 & & & w_{11} & w_{10} & \cdots & w_7 & \\ \hline
\text{Sum} & p_0 & p_1 & \cdots & p_5 & p_6 & & r_1 & r_2 & \cdots & r_5 & r_6 \\
\text{Diff.} & q_0 & q_1 & \cdots & q_5 & & & s_1 & s_2 & \cdots & s_5 &
\end{array}$$

$$\begin{array}{llllllll}
 & p_0 & p_1 & p_2 & p_3 & \quad & s_1 & s_2 & s_3 \\
 & p_6 & p_5 & p_4 & & & s_5 & s_4 & \\ \hline
\text{Sum} & l_0 & l_1 & l_2 & l_3 & & k_1 & k_2 & k_3 \\
\text{Diff.} & m_0 & m_1 & m_2 & & & n_1 & n_2 &
\end{array}$$

$$\begin{array}{ll}
l_0 & l_1 \\
l_2 & l_3 \\ \hline
g_0 & g_1
\end{array}
\qquad
\begin{aligned} q_0 - q_4 &= t_0 \\ q_1 - q_3 - q_5 &= t_1 \end{aligned}
\qquad
\begin{aligned} r_1 + r_3 - r_5 &= u_1 \\ r_2 - r_6 &= u_2 \end{aligned}$$

Multipliers	Cosine terms				Sine terms		
sin 30° = 0.5 sin 60° = 0.866 sin 90° = 1.0	g_0 g_1	m_2 m_1 m_0	$-l_2$ l_1 l_0 $-l_2$	m_0 m_2	k_1 k_2 k_3	n_1 n_2	k_1 k_3
Sum of 1st col. Sum of 2d col.	....	....	....	....	....	....	....
Sum Difference	$24\,a_0$ $24\,a_{12}$	$12\,a_2$ $12\,a_{10}$	$12\,a_4$ $12\,a_8$	$12\,a_6$	$12\,b_2$ $12\,b_{10}$	$12\,b_4$ $12\,b_8$	$12\,b_6$

Multipliers	Cosine terms						Sine terms					
$\sin 15° = 0.259$		q_5				q_1	r_1				r_5	
$\sin 30° = 0.5$	q_4				q_4			r_2				r_2
$\sin 45° = 0.707$		q_3		t_1		$-q_3$	r_3		u_1		$-r_3$	
$\sin 60° = 0.866$	q_2				$-q_2$			r_4				$-r_4$
$\sin 75° = 0.966$		q_1				q_5	r_5				r_1	
$\sin 90° = 1.0$	q_0		t_0		q_0			r_6		u_2		r_6
Sum of 1st col.	....		....		....		....		....		....	
Sum of 2d col.	....		....		....		....		....		....	
Sum	$12\,a_1$		$12\,a_3$		$12\,a_5$		$12\,b_1$		$12\,b_3$		$12\,b_5$	
Difference	$12\,a_{11}$		$12\,a_9$		$12\,a_7$		$12\,b_{11}$		$12\,b_9$		$12\,b_7$	

Checks: $y_0 = a_0 + a_1 + a_2 + \ldots + a_{12}.$

$$\tfrac{1}{2}(y_1 - y_{23}) = 0.259\,(b_1 + b_{11}) + \tfrac{1}{2}(b_2 + b_{10}) + 0.707\,(b_3 + b_9) + 0.866\,(b_4 + b_8) + 0.966\,(b_5 + b_7) + b_6.$$

Result:

$$y = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_{12} \cos 12x + b_1 \sin x + b_2 \sin 2x + \ldots + b_{11} \sin 11x,$$

or $y = c_0 + c_1 \sin(x + \phi_1) + c_2 \sin(2x + \phi_2) + \cdots + c_{12} \sin(12x + \phi_{12}).$

We shall now pass on to the evaluation of the coefficients when only the odd harmonics are present.*

90. Numerical evaluation of the coefficients. Odd harmonics only. — Most problems in alternating currents and voltages present waves where the second half-period is merely a repetition below the axis of the first half-period; the axis or zero line is chosen midway between the highest and lowest points of the wave (Fig. 86*c*). We have shown in Art. **86** that, in such cases, the trigonometric series contains only the odd harmonics. Furthermore, since the sum of the ordinates over the entire period is evidently zero, then $a_0 = \frac{1}{n}\sum y = 0$, and the series does not contain the constant term a_0. Again, since

$$\cos k\,(x + \pi) = \cos(kx + k\pi) = -\cos kx, \text{ when } k \text{ is odd},$$
$$\sin k\,(x + \pi) = \sin(kx + k\pi) = -\sin kx, \text{ when } k \text{ is odd},$$

and $\quad y_{x+\pi} = -y_x, \qquad \therefore \quad y_{x+\pi} \cos k\,(x + \pi) = y_x \cos kx,$

and $\sum y \cos kx$ has the same value over the second half-period as over the

* T. R. Running, Empirical Formulas, p. 74, gives similar schemes with 8, 10, 16, and 20 ordinates, for waves having even and odd harmonics. H. O. Taylor, in the Physical Review, N. S., Vol. VI (1915), p. 303, gives a somewhat different scheme with 24 ordinates for waves having even and odd harmonics. A very convenient computing form for the above scheme with 24 ordinates has been devised by E. T. Whittaker for use in his mathematical laboratory at the University of Edinburgh; see Carse and Shearer, *ibid.*, p. 22.

first half. Hence in finding the coefficients we need merely carry the summation over the first half-period; thus,

$$a_k = \frac{2}{n}\sum y \cos kx, \quad b_k = \frac{2}{n}\sum y \sin kx,$$

where k is odd, x and y are measured in the first half-period only, and n is the number of intervals into which the half-period is divided.

(I) *Odd harmonics up to the fifth.*—Given the curve and wishing to determine the coefficients in the equation

$$y = a_1 \cos x + a_3 \cos 3\,x + a_5 \cos 5\,x + b_1 \sin x + b_3 \sin 3\,x + b_5 \sin 5\,x,$$

we choose the origin where the wave crosses the axis, so that when $x_0 = 0$, $y_0 = 0$, divide the half-period into 6 equal parts, and measure the 5 ordinates y_1, y_2, y_3, y_4, y_5. Thus we have

x	$30°$	$60°$	$90°$	$120°$	$150°$
y	y_1	y_2	y_3	y_4	y_5

For the coefficients we have the following equations:

$$\begin{aligned}
3\,a_1 &= y_1 \cos 30° + y_2 \cos 60° + y_3 \cos 90° + y_4 \cos 120° + y_5 \cos 150°.\\
3\,a_3 &= y_1 \cos 90° + y_2 \cos 180° + y_3 \cos 270° + y_4 \cos 360° + y_5 \cos 450°.\\
3\,a_5 &= y_1 \cos 150° + y_2 \cos 300° + y_3 \cos 450° + y_4 \cos 600° + y_5 \cos 750°.\\
3\,b_1 &= y_1 \sin 30° + y_2 \sin 60° + y_3 \sin 90° + y_4 \sin 120° + y_5 \sin 150°.\\
3\,b_3 &= y_1 \sin 90° + y_2 \sin 180° + y_3 \sin 270° + y_4 \sin 360° + y_5 \sin 450°.\\
3\,b_5 &= y_1 \sin 150° + y_2 \sin 300° + y_3 \sin 450° + y_4 \sin 600° + y_5 \sin 750°.
\end{aligned}$$

Simplifying and replacing the trigonometric functions by their values in terms of sin 30° and sin 60°, we may write

$$\begin{aligned}
3\,a_1 &= (y_2 - y_4) \sin 30° + (y_1 - y_5) \sin 60°.\\
3\,a_3 &= -(y_2 - y_4) \sin 90°.\\
3\,a_5 &= (y_2 - y_4) \sin 30° - (y_1 - y_5) \sin 60°.\\
3\,b_1 &= (y_1 + y_5) \sin 30° + (y_2 + y_4) \sin 60° + y_3 \sin 90°.\\
3\,b_3 &= (y_1 - y_3 + y_5) \sin 90°.\\
3\,b_5 &= (y_1 + y_5) \sin 30° - (y_2 + y_4) \sin 60° + y_3 \sin 90°.
\end{aligned}$$

We may conveniently arrange the work in the following computing form:

$$\begin{array}{lccc}
 & y_1 & y_2 & y_3\\
 & y_5 & y_4 & \\
\hline
\text{Sum} & s_1 & s_2 & s_3\\
\text{Diff.} & d_1 & d_2 &
\end{array}$$

Checks: $0 = a_1 + a_3 + a_5$.
$y_3 = b_1 - b_3 + b_5$.

Multipliers	Cosine terms		Sine terms	
sin 30° = 0.5	d_2		s_1	
sin 60° = 0.866	d_1		s_2	
sin 90° = 1.0		$-d_2$	s_3	s_1 s_3
Sum of 1st col.	......		......	......
Sum of 2d col.	......		......	......
Sum	$3\,a_1$	$3\,a_3$	$3\,b_1$	
Diff.	$3\,a_5$		$3\,b_5$	$3\,b_3$

Result:

$$y = a_1 \cos x + a_3 \cos 3\,x + a_5 \cos 5\,x + b_1 \sin x + b_3 \sin 3\,x + b_5 \sin 5\,x.$$

The following example will illustrate the rapidity with which the coefficients may be determined.

Example. We wish to analyze the symmetric wave of Fig. 90*a*, *i.e.*, to find the coefficients of the 1st, 3d, and 5th harmonics. Choose the x-axis midway between the highest and lowest points of the wave, and

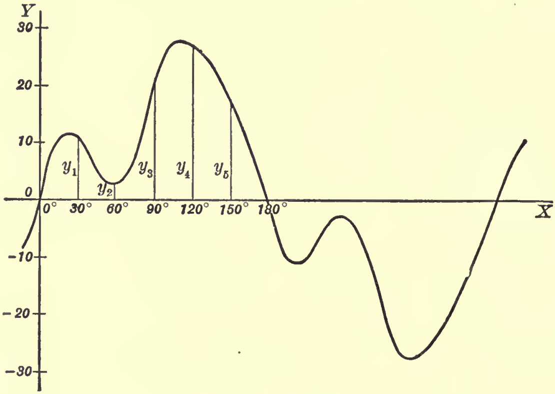


FIG. 90*a*.

the origin at the point where the wave crosses this axis in the positive direction. Then divide the half-period into 6 equal parts and measure the ordinates $y_1, \ldots, y_5$. These are given in the following table:

x	30°	60°	90°	120°	150°
y	10.7	2.8	20.5	26.5	16.6

We arrange the work in the above computing form.

	10.7	2.8	20.5
	16.6	26.5	
Sum (s)	27.3	29.3	20.5
Diff. (d)	−5.9	−23.7	

Multipliers	Cosine terms		Sine terms	
0.5	−11.85		13.65	
0.866	−5.11		25.37	
1.0		23.7	20.5	27.3 20.5
Sum of 1st col.	−11.85		34.15	27.3
Sum of 2d col.	−5.11		25.37	20.5
Sum	−16.96	23.7	59.52	
Diff.	−6.74		8.78	6.8
Divide by 3	$a_1 = -5.65$ $a_5 = -2.25$	$a_3 = 7.9$	$b_1 = 19.84$ $b_5 = 2.93$	$b_3 = 2.27$

Check: $a_1 + a_3 + a_5 = -5.65 + 7.90 - 2.25 = 0.$

$b_1 - b_3 + b_5 = 19.84 - 2.27 + 2.93 = 20.5 = y_3.$

Result:

$$y = -5.65 \cos x + 7.90 \cos 3x - 2.25 \cos 5x$$
$$+ 19.84 \sin x + 2.27 \sin 3x + 2.93 \sin 5x.$$

(II) *Odd harmonics up to the eleventh.*—Given a symmetric curve and wishing to determine the coefficients in the equation

$$y = a_1 \cos x + a_3 \cos 3x + \cdots + a_{11} \cos 11x$$
$$+ b_1 \sin x + b_3 \sin 3x + \cdots + b_{11} \sin 11x,$$

we choose the origin at the point where the wave crosses the axis, so that $y_0 = 0$, divide the half-period into 12 equal parts, and measure the 11 ordinates $y_1, y_2, \ldots, y_{11}$. Thus we have

x	$15°$	$30°$	$45°$	$\ldots$	$165°$
y	y_1	y_2	y_3	$\ldots$	y_{11}

For the coefficients we have the following equations:

$$6a_1 = y_1 \cos 15° + y_2 \cos 30° + \cdots + y_{11} \cos 165°.$$
$$6a_3 = y_1 \cos 45° + y_2 \cos 90° + \cdots + y_{11} \cos 495°.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$6b_1 = y_1 \sin 15° + y_2 \sin 30° + \cdots + y_{11} \sin 165°.$$
$$6b_3 = y_1 \sin 45° + y_2 \sin 90° + \cdots + y_{11} \sin 495°.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

If we arrange the ordinates in two rows,

	y_1	y_2	y_3	y_4	y_5	y_6
	y_{11}	y_{10}	y_9	y_8	y_7	
Sum	s_1	s_2	s_3	s_4	s_5	s_6
Diff.	d_1	d_2	d_3	d_4	d_5	d_6

replace the trigonometric functions by their values in terms of the sines of 15°, 30°, 45°, 60°, 75°, 90°, and collect terms, we may write

$$6a_1 = d_5 \sin 15° + d_4 \sin 30° + d_3 \sin 45° + d_2 \sin 60° + d_1 \sin 75°.$$
$$6a_{11} = -d_5 \sin 15° + d_4 \sin 30° - d_3 \sin 45° + d_2 \sin 60° - d_1 \sin 75°.$$
$$6a_5 = d_1 \sin 15° + d_4 \sin 30° - d_3 \sin 45° - d_2 \sin 60° + d_5 \sin 75°.$$
$$6a_7 = -d_1 \sin 15° + d_4 \sin 30° + d_3 \sin 45° - d_2 \sin 60° - d_5 \sin 75°.$$
$$6b_1 = s_1 \sin 15° + s_2 \sin 30° + s_3 \sin 45° + s_4 \sin 60° + s_5 \sin 75° + s_6 \sin 90°.$$
$$6b_{11} = s_1 \sin 15° - s_2 \sin 30° + s_3 \sin 45° - s_4 \sin 60° + s_5 \sin 75° - s_6 \sin 90°.$$
$$6b_5 = s_5 \sin 15° + s_2 \sin 30° - s_3 \sin 45° - s_4 \sin 60° + s_1 \sin 75° + s_6 \sin 90°.$$
$$6b_7 = s_5 \sin 15° - s_2 \sin 30° - s_3 \sin 45° + s_4 \sin 60° + s_1 \sin 75° - s_6 \sin 90°.$$
$$6a_3 = (d_1 - d_3 - d_5) \sin 45° - d_4 \sin 90°.$$
$$6a_9 = -(d_1 - d_3 - d_5) \sin 45° - d_4 \sin 90°.$$
$$6b_3 = (s_1 + s_3 - s_5) \sin 45° + (s_2 - s_6) \sin 90°.$$
$$6b_9 = (s_1 + s_3 - s_5) \sin 45° - (s_2 - s_6) \sin 90°.$$

We may conveniently arrange the work in the following computing form:

	y_1	y_2	y_3	y_4	y_5	y_6
	y_{11}	y_{10}	y_9	y_8	y_7	
Sum	s_1	s_2	s_3	s_4	s_5	s_6
Diff.	d_1	d_2	d_3	d_4	d_5	

$$
\begin{aligned}
s_1 + s_3 - s_5 &= r_1\\
s_2 - s_6 &= r_2\\
d_1 - d_3 - d_5 &= e_1
\end{aligned}
$$

Multipliers	Cosine terms						Sine terms					
sin 15° = 0.259		d_5				d_1	s_1				s_5	
sin 30° = 0.5	d_4				d_4			s_2				s_2
sin 45° = 0.707		d_3		e_1		$-d_3$	s_3		r_1		$-s_3$	
sin 60° = 0.866	d_2				$-d_2$			s_4				$-s_4$
sin 75° = 0.966		d_1				d_5	s_5				s_1	
sin 90° = 1.0			$-d_4$					s_6		r_2		s_6
Sum of 1st col.	....		....		....		....		....		....	
Sum of 2d col.	....		....		....		....		....		....	
Sum	$6\,a_1$		$6\,a_3$		$6\,a_5$		$6\,b_1$		$6\,b_3$		$6\,b_5$	
Diff.	$6\,a_{11}$		$6\,a_9$		$6\,a_7$		$6\,b_{11}$		$6\,b_9$		$6\,b_7$	

Checks:
$$
\begin{aligned}
a_1 + a_3 + a_5 + a_7 + a_9 + a_{11} &= 0,\\
b_1 - b_3 + b_5 - b_7 + b_9 - b_{11} &= y_6.
\end{aligned}
$$

Result:
$$
\begin{aligned}
y = {}& a_1 \cos x + a_3 \cos 3\,x + \cdots + a_{11} \cos 11\,x\\
&+ b_1 \sin x + b_3 \sin 3\,x + \cdots + b_{11} \sin 11\,x.
\end{aligned}
$$

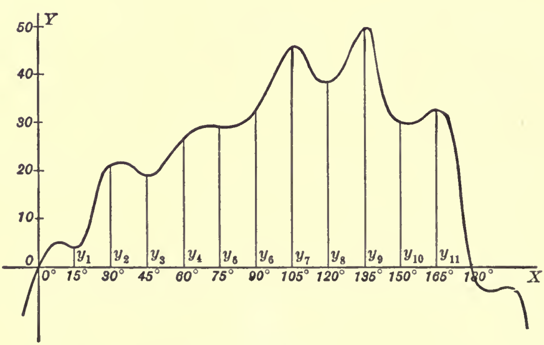


FIG. 90*b*.

Example. Fig. 90*b* represents a half-period of an e.m.f. wave whose frequency is 60 cycles. We wish to find the odd harmonics up to the 11th order. Choose the x-axis midway between the highest and lowest points of the complete wave and the origin at the point where the wave crosses the x-axis in the positive direction. Divide the half-period into 12 equal

parts and measure the ordinates $y_1, y_2 \ldots, y_{11}$. These are given in the following table:

x	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°
y	4	21	19	27	29	33	46	38	50	30	33

We arrange the work in the above computing form.

	4	21	19	27	29	33	$37 + 69 - 75 = 31 = r_1$
	33	30	50	38	46		$51 - 33 = 18 = r_2$
Sums (s)	37	51	69	65	75	33	$-29 + 31 + 17 = 19 = e_1$
Diff. (d)	−29	−9	−31	−11	−17		

Multipliers	Cosine terms			Sine terms		
0.259	−4.4		−7.5	9.6		19.4
0.5	−5.5		−5.5	25.5		25.5
0.707	−21.9	13.4	21.9	48.8	21.9	−48.8
0.866	−7.8		7.8	56.3		−56.3
0.966	−28.0		−16.4	72.5		35.7
1.0		11.0		33.0	18.0	33.0
Sum 1st col.	−13.3	11.0	2.3	130.9	21.9	6.3
Sum 2d col.	−54.3	13.4	−2.0	114.8	18.0	2.2
Sum	−67.6	24.4	0.3	245.7	39.9	8.5
Diff.	41.0	−2.4	4.3	16.1	3.9	4.1
Divide by 6	$a_1 = -11.27$	$a_3 = 4.07$	$a_5 = 0.05$	$b_1 = 40.95$	$b_3 = 6.65$	$b_5 = 1.42$
	$a_{11} = 6.83$	$a_9 = -0.40$	$a_7 = 0.72$	$b_{11} = 2.68$	$b_9 = 0.65$	$b_7 = 0.68$

Check:

$$a_1 + a_3 + \cdots + a_{11} = -11.27 + 4.07 + 0.05 + 0.72 - 0.40 + 6.83 = 0,$$

$$b_1 - b_3 + \cdots - b_{11} = 40.95 - 6.65 + 1.42 - 0.68 + 0.65 - 2.68 = 33.01 = y_6.$$

Result:

$$y = -11.27 \cos x + 4.07 \cos 3x + 0.05 \cos 5x + 0.72 \cos 7x - 0.40 \cos 9x + 6.83 \cos 11x + 40.95 \sin x + 6.65 \sin 3x + 1.42 \sin 5x + 0.68 \sin 7x + 0.65 \sin 9x + 2.68 \sin 11x.$$

(III) *Odd harmonics up to the seventeenth.* — Given a symmetric curve and wishing to determine the coefficients in the equation

$$y = a_1 \cos x + a_3 \cos 3x + \cdots + a_{17} \cos 17x + b_1 \sin x + b_3 \sin 3x + \cdots + b_{17} \sin 17x,$$

we choose the origin at the point where the wave crosses the axis, so that $y_0 = 0$, divide the half-period into 18 equal parts, and measure the 17 ordinates $y_1, y_2, \ldots, y_{17}$. Thus we have

x	10°	20°	30°	. . .	170°
y	y_1	y_2	y_3	. . .	y_{17}

If we use the same method as that employed in deriving the 11-ordinate scheme, we shall arrive at the following 17-ordinate computing form. This form is self-explanatory.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
	y_{17}	y_{16}	y_{15}	y_{14}	y_{13}	y_{12}	y_{11}	y_{10}	
Sum	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
Diff.	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	

	s_1	s_2	s_3	r_1	d_2	d_1	$-d_6$	$-e_1$
	s_5	s_4	$-s_9$	$-r_3$	$-d_4$	$-d_5$		e_3
	$-s_7$	$-s_8$			$-d_8$	$-d_7$		
Sum	r_1	r_2	r_3	r_4	e_1	e_2	e_3	e_4

Multipliers	Cosine terms					Sine terms				
sin 10° = 0.1737	d_8		$-d_2$	d_4		s_1		$-s_7$	$-s_5$	
sin 20° = 0.3420	d_7		$-d_5$	d_1		s_2		$-s_4$	$-s_8$	
sin 30° = 0.5000	d_6	e_1	d_6	d_6		s_3	r_1	s_3	$-s_3$	
sin 40° = 0.6428	d_5		d_1	$-d_7$		s_4		s_8	s_2	
sin 50° = 0.7660	d_4		d_8	$-d_2$		s_5		s_1	s_7	
sin 60° = 0.8660	d_3	e_2	$-d_3$	$-d_3$		s_6	r_2	$-s_6$	s_6	
sin 70° = 0.9397	d_2		$-d_4$	$-d_8$		s_7		$-s_5$	s_1	
sin 80° = 0.9848	d_1		d_7	d_5		s_8		s_2	$-s_4$	
sin 90° = 1.0000		e_3			e_4	s_9	r_3	s_9	$-s_9$	r_4
Sum of 1st col.	.....	......	....	....		......	.....	....	.....	
Sum of 2d col.	.....	.....	....	....		.....	. .	. . .	. .	
Sum	$9\,a_1$	$9\,a_3$	$9\,a_5$	$9\,a_7$	$9\,a_9$	$9\,b_1$	$9\,b_3$	$9\,b_5$	$9\,b_7$	$9\,b_9$
Diff.	$9\,a_{17}$	$9\,a_{15}$	$9\,a_{13}$	$9\,a_{11}$		$9\,b_{17}$	$9\,b_{15}$	$9\,b_{13}$	$9\,b_{11}$	

Check:
$$a_1 + a_3 + a_5 + \cdots + a_{17} = 0,$$
$$b_1 - b_3 + b_5 - \cdots + b_{17} = y_9.$$

Result:
$$y = a_1 \cos x + a_3 \cos 3\,x + \cdots + a_{17} \cos 17\,x$$
$$+ b_1 \sin x + b_3 \sin 3\,x + \cdots + b_{17} \sin 11\,x.$$

Similar computing forms may be constructed for symmetrical waves containing odd harmonics up to the seventh, ninth, etc., orders.

91. Numerical evaluation of the coefficients. Averaging selected ordinates.* — We are to determine the coefficients in the trigonometric series

$$y = a_0 + a_1 \cos x + a_2 \cos 2\,x + \cdots + a_k \cos kx + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2\,x + \cdots + b_k \sin kx + \cdots.$$

Let a_n and b_n represent the coefficients of any harmonic. We divide the period $2\,\pi$ into n equal intervals of width $2\,\pi/n$ and measure the ordinates at the beginning of these intervals. We have the table

x	x_0	x_1	x_2	$\cdots$	x_r	$\cdots$	x_{n-1}
y	y_0	y_1	y_2	$\cdots$	y_r	$\cdots$	y_{n-1}

* These methods have been developed by J. Fischer-Hinnen, Elekrotechnische Zeitschrift, May 9, 1901, and S. P. Thompson, Proc. of the Phys. Soc. of London, Vol. XXIII, 1911, p. 334. See, also, a description of the Fischer-Hinnen method by P. M. Lincoln, The Electric Journal, Vol. 5, 1908, p. 386.

Substituting these pairs of values in our series, we have n equations of the form

$$y_r = a_0 + a_1 \cos x_r + a_2 \cos 2x_r + \cdots + a_k \cos kx_r + \cdots$$
$$+ b_1 \sin x_r + b_2 \sin 2x_r + \cdots + b_r \sin kx_r + \cdots,$$

where r takes in succession the values 0, 1, 2, 3, . . . , $n - 1$; adding these n equations, we get

$$\sum y_r = na_0 + a_1 \sum \cos x_r + \cdots + a_k \sum \cos kx_r + \cdots + b_k \sum \sin kx_r + \cdots,$$

where the summation is carried from $r = 0$ to $r = n - 1$.

If we let $\beta = k\dfrac{2\pi}{n}$ in the expressions for $\sum \cos(\alpha + r\beta)$ and $\sum \sin(\alpha + r\beta)$ derived in the note on p. 175, these become

$$\sum \cos\left(\alpha + kr\frac{2\pi}{n}\right) = \frac{\sin k\pi}{\sin(k\pi/n)} \cos\left(\alpha + \frac{k(n-1)\pi}{n}\right) = 0, \text{ since } \sin k\pi = 0,$$

$$\sum \sin\left(\alpha + kr\frac{2\pi}{n}\right) = \frac{\sin k\pi}{\sin(k\pi/n)} \sin\left(\alpha + \frac{k(n-1)\pi}{n}\right) = 0, \text{ since } \sin k\pi = 0,$$

except when k is a multiple of n, for then both $\sin k\pi$ and $\sin(k\pi/n)$ are equal to zero and the fractional expression becomes indeterminate But when k is a multiple of n,

$$\sum \cos\left(\alpha + kr\frac{2\pi}{n}\right) = \sum \cos(\alpha + \text{multiple of } 2\pi) = \sum \cos\alpha = n\cos\alpha.$$

$$\sum \sin\left(\alpha + kr\frac{2\pi}{n}\right) = \sum \sin(\alpha + \text{multiple of } 2\pi) = \sum \sin\alpha = n\sin\alpha.$$

Hence we may state

$$\sum \cos\left(\alpha + kr\frac{2\pi}{n}\right) = 0, \text{ except when } k = n, 2n, 3n, \ldots$$
$$= n\cos\alpha, \text{ when } k = n, 2n, 3n, \ldots.$$
$$\sum \sin\left(\alpha + kr\frac{2\pi}{n}\right) = 0, \text{ except when } k = n, 2n, 3n, \ldots$$
$$= n\sin\alpha, \text{ when } k = n, 2n, 3n, \ldots.$$

(1) If we start our intervals at $x_0 = 0$, then $x_r = r\dfrac{2\pi}{n}$, and

$$\sum \cos kx_r = \sum \cos\left(0 + kr\frac{2\pi}{n}\right) = 0, \text{ except when } k = n, 2n, 3n, \ldots$$
$$= n\cos 0 = n, \text{ when } k = n, 2n, 3n, \ldots.$$
$$\sum \sin kx_r = \sum \sin\left(0 + kr\frac{2\pi}{n}\right) = 0, \text{ for all values of } k.$$

$$\therefore \quad \sum y_r = na_0 + na_n + na_{2n} + \cdots = n(a_0 + a_n + a_{2n} + a_{3n} + \cdots).$$

(2) If we start our intervals at $x_0' = \frac{\pi}{n}$, then $x_r' = \frac{\pi}{n} + r\frac{2\pi}{n}$ and

$$\sum \cos kx_r' = \sum \cos\left(k\frac{\pi}{n} + kr\frac{2\pi}{n}\right) = 0 \text{ except when } k = n,\ 2n,\ 3n,\ \ldots$$

$$= n\cos\frac{k\pi}{n}\begin{cases} = n \text{ when } k = 2n,\ 4n,\ 6n,\ \ldots \\ = -n \text{ when } k = n,\ 3n,\ 5n,\ \ldots \end{cases}$$

$$\sum \sin kx_r' = \sum \sin\left(k\frac{\pi}{n} + kr\frac{2\pi}{n}\right) = 0 \text{ for all values of } k.$$

$$\therefore \quad \sum y_r' = na_0 - na_n + na_{2n} - na_{3n} + \cdots$$
$$= n(a_0 - a_n + a_{2n} - a_{3n} + a_{4n} - \cdots).$$

Subtracting the second set of ordinates, y', from the first set, y, we have

$$\sum y_r - \sum y_r' = \sum (y_r - y_r') = y_0 - y_0' + y_1 - y_1' + y_2 - y_2' + \cdots + y_{n-1} - y'_{n-1}$$
$$= 2n(a_n + a_{3n} + a_{5n} + \cdots),$$

or $\quad a_n + a_{3n} + a_{5n} + \cdots = \frac{1}{2n}(y_0 - y_0' + y_1 - y_1' + \cdots + y_{n-1} - y'_{n-1}).$

The first set of n ordinates start at $x = 0$ and are at intervals of $2\pi/n$, and the second set of n ordinates, start at $x = \pi/n$ and are at intervals of $2\pi/n$; thus, the period from $x = 0$ to $x = 2\pi$ is divided into $2n$ equal parts each of width π/n (Fig. 91*a*). Hence,

If, starting at $x = 0$, we measure $2n$ ordinates at intervals of π/n, the average of these ordinates taken alternately plus and minus is equal to the sum of the amplitudes of the nth, 3 nth, 5 nth, . . . cosine components.

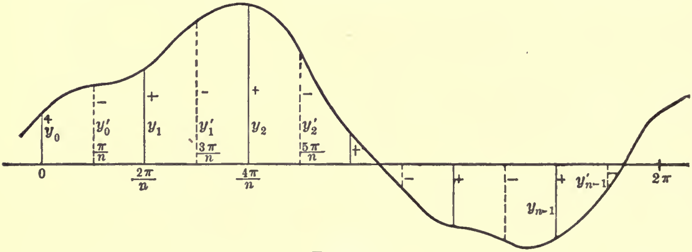


FIG. 91*a*.

Thus, to determine the sum of the amplitudes of the 5th, 15th, 25th, . . . cosine components, merely average the 10 ordinates, taken alternately plus and minus, at intervals of $180° \div 5 = 36°$, or at $0°$, $36°$, $72°$, . . . , $324°$ (Fig. 91*c*); therefore

$$a_5 + a_{15} + a_{25} + \cdots = \tfrac{1}{10}(y_0 - y_{36} + y_{72} - y_{108} + y_{144} - y_{180} + y_{216} - y_{252} + y_{288} - y_{324}).$$

If the 15th, 25th, . . . harmonics are not present, then

$$a_5 = \tfrac{1}{10}(y_0 - y_{36} + y_{72} - y_{108} + y_{144} - y_{180} + y_{216} - y_{252} + y_{288} - y_{324}).$$

(3) Similarly, if we start our intervals at $\bar{x}_0 = \dfrac{\pi}{2n}$, then $\bar{x}_r = \dfrac{\pi}{2n} + r\dfrac{2\pi}{n}$, and

$$\sum \cos k\bar{x}_r = \sum \cos\left(k\frac{\pi}{2n} + kr\frac{2\pi}{n}\right) = 0 \text{ for all values of } k,$$

$$\sum \sin k\bar{x}_r = \sum \sin\left(k\frac{\pi}{2n} + kr\frac{2\pi}{n}\right) = 0 \text{ except when } k = n, 2n, 3n, \ldots$$

$$= n\sin\frac{k\pi}{2n}\begin{cases} = n \text{ when } k = n, 5n, 9n, \ldots \\ = 0 \text{ when } k = 2n, 4n, 6n, \ldots \\ = -n \text{ when } k = 3n, 7n, 11n, \ldots \end{cases}$$

$$\therefore \sum \bar{y}_r = na_0 + nb_n - nb_{3n} + nb_{5n} - \cdots = n(a_0 + b_n - b_{3n} + b_{5n} - b_{7n} + \cdots).$$

(4) Again, if we start our intervals at $\bar{x}_0' = \dfrac{\pi}{2n} + \dfrac{\pi}{n}$, then $\bar{x}_r' = \dfrac{3\pi}{2n} + r\dfrac{2\pi}{n}$, and

$$\sum \cos k\bar{x}_r' = \sum \cos\left(k\frac{3\pi}{2n} + kr\frac{2\pi}{n}\right) = 0 \text{ for all values of } k,$$

$$\sum \sin k\bar{x}_r' = \sum \sin\left(k\frac{3\pi}{2n} + kr\frac{2\pi}{n}\right) = 0 \text{ except when } k = n, 2n, 3n, \ldots$$

$$= n\sin\frac{3k\pi}{2n}\begin{cases} = -n \text{ when } k = n, 5n, 9n, \ldots \\ = 0 \text{ when } k = 2n, 4n, 6n, \ldots \\ = n \text{ when } k = 3n, 7n, 11n, \ldots \end{cases}$$

$$\therefore \sum \bar{y}_r' = na_0 - nb_n + nb_{3n} - nb_{5n} + \cdots = n(a_0 - b_n + b_{3n} - b_{5n} + b_{7n} - \cdots).$$

Subtracting the second set of ordinates, $\bar{y}'$, from the first set, $\bar{y}$, we have

$$\sum \bar{y}_r - \sum \bar{y}_r' = \sum(\bar{y} - \bar{y}_r') = \bar{y}_0 - \bar{y}_0' + \bar{y}_1 - \bar{y}_1' + \cdots + \bar{y}_{n-1} - \bar{y}'_{n-1}$$
$$= 2n(b_n - b_{3n} + b_{5n} - b_{7n} + \cdots),$$

or $b_n - b_{3n} + b_{5n} - b_{7n} + \cdots = \dfrac{1}{2n}(\bar{y}_0 - \bar{y}_0' + \bar{y}_1 - \bar{y}_1' + \cdots + \bar{y}_{n-1} - \bar{y}'_{n-1}).$

The first set of n ordinates start at $x = \pi/2n$ and are at intervals of $2\pi/n$, and the second set of n ordinates start at $x = \dfrac{\pi}{2n} + \dfrac{\pi}{n}$ and are at intervals of $2\pi/n$; thus the period from $x = \pi/2n$ to $x = 2\pi + \pi/2n$ is divided into $2n$ equal parts each of width $\dfrac{\pi}{n}$. Hence,

If, starting at $x = \pi/2n$, we measure $2n$ ordinates at intervals of π/n, the average of these ordinates taken alternately plus and minus is equal to the

sum of the amplitudes, taken alternately plus and minus, of the nth, 3 nth, 5 nth, . . . sine components.

Thus to determine the sum of the amplitudes, taken alternately plus and minus, of the 5th, 15th, 25th, . . . sine components, merely average the 10 ordinates taken alternately plus and minus, at intervals of $180° \div 5 = 36°$, starting at $x = 180° \div 10 = 18°$, *i.e.*, at $x = 18°$, $54°, 90°, \ldots, 342°$ (Fig. 91*c*); therefore

$$b_5 - b_{15} + b_{25} - \cdots = \tfrac{1}{10}(y_{18} - y_{54} + y_{90} - y_{126} + y_{162} - y_{198} + y_{234} - y_{270} + y_{306} - y_{342}).$$

If the 15th, 25th, . . . harmonics are not present, then

$$b_5 = \tfrac{1}{10}(y_{18} - y_{54} + y_{90} - y_{126} + y_{162} - y_{198} + y_{234} - y_{270} + y_{306} - y_{342}).$$

We may also note that the set of $2n$ ordinates measured for determining the b's lie midway between the set of $2n$ ordinates measured for determining the a's, so that to determine any desired harmonic we actually measure $4n$ ordinates, starting at $x = 0$ and at intervals of $\pi/2n$. We use the 1st, 3d, 5th, . . . of these ordinates for determining a, and the 2d, 4th, 6th, . . . of these ordinates for determining b.

If the higher harmonics are present, these must be evaluated first. The absolute term a_0 is obtained from the relation

$$y_0 = a_0 + a_1 + a_2 + a_3 + \cdots.$$

We shall now illustrate the methods developed by an example.

Example. Given the periodic wave of Fig. 89 and assuming that no higher harmonics than the 6th are present, we are to determine the coefficients in the equation

$$y = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_6 \cos 6x + b_1 \sin x + b_2 \sin 2x + \cdots + b_6 \sin 6x.$$

To determine a_6 and b_6 measure 12 ordinates at intervals of 30° beginning at $x = 0°$ and $x = 15°$ respectively (Fig. 91*b*); then

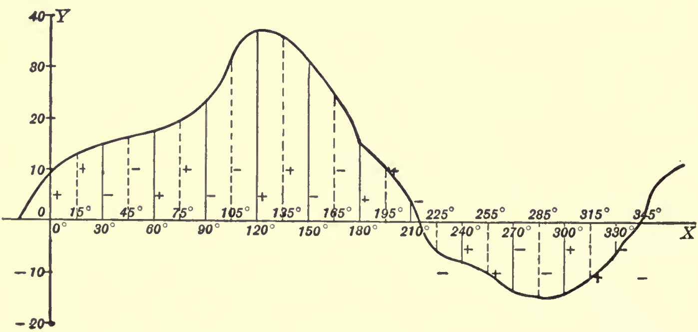


FIG. 91*b*.

$$a_6 = \tfrac{1}{12}(y_0 - y_{30} + y_{60} - y_{90} + \cdots + y_{300} - y_{330})$$
$$= \tfrac{1}{12}(9.3 - 15.0 + 17.4 - 23.0 + 37.0 - 31.0 + 15.3 - 4.0 - 8.0 + 13.2 - 14.2 + 6.0) = 0.25.$$
$$b_6 = \tfrac{1}{12}(y_{15} - y_{45} + y_{75} - y_{105} + \cdots + y_{315} - y_{345})$$
$$= \tfrac{1}{12}(13.0 - 16.0 + 19.5 - 31.0 + 35.3 - 23.8 + 10.5 + 5.7 - 10.0 + 14.5 - 11.0 - 0.5) = 0.52.$$

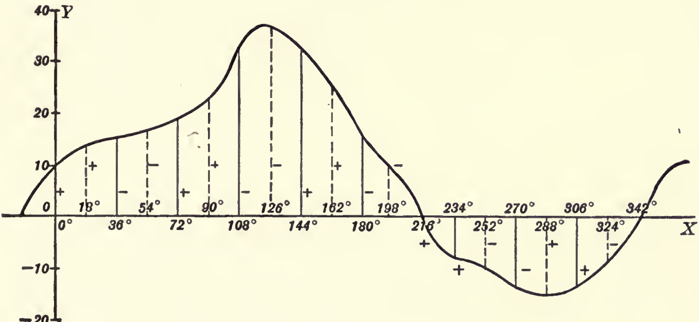


FIG. 91c.

To determine a_5 and b_5 measure 10 ordinates at intervals of 36°, beginning at $x = 0°$ and $x = 18°$ respectively (Fig. 91c) then

$$a_5 = \tfrac{1}{10}(y_0 - y_{36} + y_{72} - y_{108} + \cdots + y_{288} - y_{324})$$
$$= \tfrac{1}{10}(9.3 - 15.3 + 18.8 - 32.8 + 33.0 - 15.3 - 1.0 + 9.5 - 15.0 + 8.4)$$
$$= -0.04.$$
$$b_5 = \tfrac{1}{10}(y_{18} - y_{54} + y_{90} - y_{126} + \cdots + y_{306} - y_{342})$$
$$= \tfrac{1}{10}(13.8 - 16.8 + 23.0 - 36.8 + 25.5 - 9.0 - 7.7 + 13.4 - 13.2 + 1.5)$$
$$= -0.63.$$

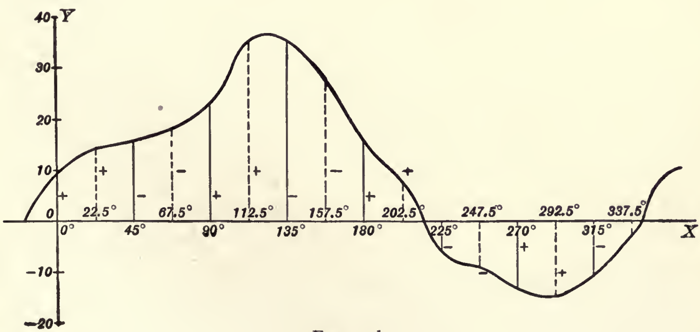


FIG. 91d.

To determine a_4 and b_4 measure 8 ordinates at intervals of 45°, beginning at $x = 0°$ and $x = 22\frac{1}{2}°$ respectively (Fig. 91d); then

$$\begin{aligned} a_4 &= \tfrac{1}{8}(y_0 - y_{45} + y_{90} - y_{135} + \cdots + y_{270} - y_{315}) \\ &= \tfrac{1}{8}(9.3 - 16.0 + 23.0 - 35.3 + 15.3 + 5.7 - 13.2 + 11.0) = -0.03. \\ b_4 &= \tfrac{1}{8}(y_{22.5} - y_{67.5} + y_{112.5} - \cdots + y_{292.5} - y_{337.5}) \\ &= \tfrac{1}{8}(14.5 - 18.0 + 35.0 - 27.7 + 7.7 + 8.8 - 14.7 + 3.0) = 1.08. \end{aligned}$$

To determine a_3 and b_3 measure 6 ordinates at intervals of 60°, beginning at $x = 0$ and $x = 30°$ respectively (Fig. 91b); then

$$\begin{aligned} a_3 &= \tfrac{1}{6}(y_0 - y_{60} + y_{120} - y_{180} + y_{240} - y_{300}) \\ &= \tfrac{1}{6}(9.3 - 17.4 + 37.0 - 15.3 - 8.0 + 14.2) = 3.30. \\ b_3 &= \tfrac{1}{6}(y_{30} - y_{90} + y_{150} - y_{210} + y_{270} - y_{330}) \\ &= \tfrac{1}{6}(15.0 - 23.0 + 31.0 - 4.0 - 13.2 + 6.0) = 1.97. \end{aligned}$$

To determine a_2 and b_2 measure 4 ordinates at intervals of 90°, beginning at $x = 0°$ and $x = 45°$ respectively (Fig. 91b); then

$$a_2 + a_6 = \tfrac{1}{4}(y_0 - y_{90} + y_{180} - y_{270}) = \tfrac{1}{4}(9.3 - 23.0 + 15.3 + 13.2) = 3.70,$$
$$\therefore \quad a_2 = 3.45.$$
$$b_2 - b_6 = \tfrac{1}{4}(y_{45} - y_{135} + y_{225} - y_{315}) = \tfrac{1}{4}(16.0 - 35.3 - 5.7 + 11.0) = -3.50,$$
$$\therefore \quad b_2 = -2.98.$$

To determine a_1 and b_1 measure 2 ordinates at intervals of 180°, beginning at $x = 0°$ and $x = 90°$ respectively (Fig. 91b); then

$$a_1 + a_3 + a_5 = \tfrac{1}{2}(y_0 - y_{180}) = \tfrac{1}{2}(9.3 - 15.3) = -3.00, \quad \therefore \quad a_1 = -6.26.$$
$$b_1 - b_3 + b_5 = \tfrac{1}{2}(y_{90} - y_{270}) = \tfrac{1}{2}(23.0 + 13.2) = 18.10, \quad \therefore \quad b_1 = 20.60.$$

To determine a_0 we have

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = y_0 = 9.3, \quad \therefore \quad a_0 = 8.63.$$

Result:

$$\begin{aligned} y = 8.63 &- 6.26 \cos x + 3.45 \cos 2x + 3.30 \cos 3x - 0.03 \cos 4x \\ &- 0.04 \cos 5x + 0.25 \cos 6x + 20.60 \sin x - 2.98 \sin 2x \\ &+ 1.97 \sin 3x + 1.08 \sin 4x - 0.63 \sin 5x + 0.52 \sin 6x. \end{aligned}$$

This result agrees quite closely with that of Art. 89, p. 184; the differences in the values of the coefficients are due to the fact that by the method of Art. 89 only the ordinates at 0°, 30°, 60°, . . . , 330° are used, whereas by the method of this Art. a large number of intermediate ordinates are used. If the curve is drawn by some mechanical instrument, the present method will evidently give better approximations to the values of the coefficients; but the labor involved in using the computing form on p. 183 is much less than that used in measuring the selected ordinates above.

92. Numerical evaluation of the coefficients. Averaging selected ordinates. Odd harmonics only. — If the axis is chosen midway between the highest and lowest points of the wave and the second half-period is

merely a repetition below the axis of the first half-period, then only the odd harmonics are present. If the ordinates at $x = x_r$ and $x = x_r + \pi$ are designated by y_r and $y_{r+\pi}$ respectively, then $y_{r+\pi} = -y_r$. In the method of averaging selected ordinates, the $2n$ ordinates are spaced at intervals of π/n and are taken alternately plus and minus; then $y_{r+\pi}$ is at a distance $\pi = n(\pi/n)$, or n intervals, from y_r, and since n is odd, $y_{r+\pi}$ will occur in the summation with sign opposite to that with which y_r occurs, so that, *e.g.*

$$a_n + a_{3n} + \cdots =$$

$$\frac{1}{2n}\left(y_0 - y_1' + \cdots \pm y_r \cdots - y_{0+\pi} + y_{1+\pi}' - \cdots \mp y_{r+\pi} \cdots\right)$$

$$= \frac{1}{2n}(2y_0 - 2y_1' + \cdots \pm 2y_r \cdots)$$

$$= \frac{1}{n}(y_0 - y_1' + \cdots \pm y_r \cdots).$$

Hence we need merely divide the half-period into n equal intervals and average n ordinates. We may therefore restate our rules for determining the coefficients if *the wave contains odd harmonics only.*

If, starting at $x = 0$, we measure n ordinates at intervals of π/n, the average of these ordinates taken alternately plus and minus is equal to the sum of the amplitudes of the nth, 3 nth, 5 nth, . . . cosine components.

If, starting at $x = \pi/2n$, we measure n ordinates at intervals of π/n, the average of these ordinates taken alternately plus and minus is equal to the sum of the amplitudes, taken alternately plus and minus, of the nth, 3 nth, 5 nth, . . . sine components.

Furthermore, $a_0 = 0$ since the sum of the ordinates over the entire period is zero.

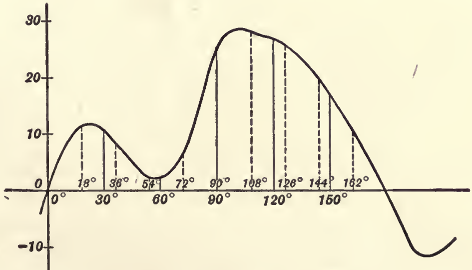


FIG. 92.

Example. Assuming that the symmetric wave of Fig. 92 contains no higher harmonics than the 5th, we are to determine the 1st, 3d, and 5th harmonics. Applying the above rules we have

$a_5 = \frac{1}{5}(y_0 - y_{36} + y_{72} - y_{108} + y_{144})$
$\quad = \frac{1}{5}(0 - 8.6 + 6.3 - 27.7 + 19.0) = -2.20.$
$b_5 = \frac{1}{5}(y_{18} - y_{54} + y_{90} - y_{126} + y_{162}) = \frac{1}{5}(11.3 - 2.7 + 20.5 - 25.5 + 10.7) = 2.86.$
$a_3 = \frac{1}{3}(y_0 - y_{60} + y_{120}) = \frac{1}{3}(0 - 2.8 + 26.5) = 7.90.$
$b_3 = \frac{1}{3}(y_{30} - y_{90} + y_{150}) = \frac{1}{3}(10.7 - 20.5 + 16.6) = 2.27.$
$a_1 + a_3 + a_5 = \frac{1}{1}(y_0) = 0, \qquad \therefore \quad a_1 = -5.70.$
$b_1 - b_3 + b_5 = \frac{1}{1}(y_{90}) = 20.5, \qquad \therefore \quad b_1 = +19.91.$

Result:

$$y = -5.70 \cos x + 7.90 \cos 3x - 2.20 \cos 5x$$
$$+ 19.91 \sin x + 2.27 \sin 3x + 2.86 \sin 5x.$$

We may compare this result with that obtained for the same curve by the use of the computing form on p. 187.

If only the 1st and 3d harmonics had been present in the above wave, we should have

$a_3 = \frac{1}{3}(y_0 - y_{60} + y_{120}); \qquad b_3 = \frac{1}{3}(y_{30} - y_{90} + y_{150});$
$a_1 + a_3 = y_0 = 0; \qquad b_1 - b_3 = y_{90}.$

If all the odd harmonics up to the ninth had been present in the above wave, we should have

$a_9 = \frac{1}{9}(y_0 - y_{20} + y_{40} - y_{60} + y_{80} - y_{100} + y_{120} - y_{140} + y_{160});$
$b_9 = \frac{1}{9}(y_{10} - y_{30} + y_{50} - y_{70} + y_{90} - y_{110} + y_{130} - y_{150} + y_{170});$
$a_7 = \frac{1}{7}(y_0 - y_{25.71} + y_{51.43} - y_{77.14} + y_{102.86} - y_{128.57} + y_{154.29});$
$b_7 = \frac{1}{7}(y_{12.86} - y_{38.57} + y_{64.29} - y_{90} + y_{115.71} - y_{141.43} + y_{167.14});$
$a_5 = \frac{1}{5}(y_0 - y_{36} + y_{72} - y_{108} + y_{144}); \; b_5 = \frac{1}{5}(y_{18} - y_{54} + y_{90} - y_{126} + y_{162});$
$a_3 + a_9 = \frac{1}{3}(y_0 - y_{60} + y_{120}); \; b_3 - b_9 = \frac{1}{3}(y_{30} - y_{90} + y_{150});$
$a_1 + a_3 + a_5 + a_7 + a_9 = y_0 = 0; \; b_1 - b_3 + b_5 - b_7 + b_9 = y_{90}.$

Similar schedules may be formed for determining the odd harmonics up to any order.

93. Graphical evaluation of the coefficients. — Various graphical methods have been devised for finding the values of the coefficients in the Fourier's series, but these are less accurate and much more laborious than the arithmetic ones. The graphical methods, while interesting, are of little practical value in rapidly analyzing a periodic curve, so that we shall describe here only one of these methods — the Ashworth-Harrison method.*

If, for example, we divide the complete period into 12 equal intervals and measure the 12 ordinates, we shall have the table

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
y	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}

* Electrician, lxvii, p. 288, 1911; Engineering, lxxxi, p. 201, 1906. Other methods are briefly mentioned and further references are given in Modern Instruments and Methods of Calculation, a handbook of the Napier Tercentenary Celebration.

We have already shown (p. 181) that

$$6\,a_1 = \sum y_r \cos x_r = y_0 \cos 0^\circ + y_1 \cos 30^\circ + \cdots + y_{11} \cos 330^\circ,$$

$$6\,b_1 = \sum y_r \sin x_r = y_0 \sin 0^\circ + y_1 \sin 30^\circ + \ldots + y_{11} \sin 330^\circ.$$

It is evident that if we consider the y's as a set of co-planar forces radiating from a common center at angles 0°, 30°, 60°, . . . , the sum of the horizontal components is equal to $6\,a_1$ and the sum of the vertical components is $6\,b_1$. To facilitate the finding of these sums we may draw the polygon of forces, starting at a point O and laying off in succession the ordinates, each making an angle of 30° with the preceding, as in Fig. 93a (proper regard must be had for the signs of the ordinates). The polygon of forces may be constructed rapidly by means of a protractor carrying an ordinary measuring scale along the diameter. Then, OA, the projection of the resultant OP on the horizontal, is equal to $6\,a_1$, and OB, the projection of the resultant OP on the vertical, is equal to $6\,b_1$. Furthermore, if we write $a_1 \cos x + b_1 \sin x = c_1 \sin (x + \phi_1)$, then the length of OP is $6\,c_1$ and the angle POB is ϕ_1. In Fig. 93a we have made the construction for the determination of a_1, b_1, c_1, and ϕ_1 for the periodic curve drawn in Fig. 89 using the table of ordinates on p. 184. We find

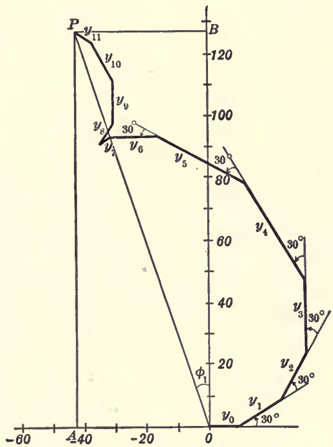


FIG. 93a.

$$OA = 6\,a_1 = -41.4, \quad OB = 6\,b_1 = 126.0, \quad OP = 6\,c_1 = 134,$$
$$\angle\, POB = \phi_1 = -18.1^\circ;$$

hence

$$a_1 = -6.9, \qquad b_1 = 21.0, \qquad c_1 = 22.3, \qquad \phi_1 = -18.1^\circ.$$

These results agree very closely with those obtained on p. 184.

We may find a_2 and b_2 by laying off in succession the ordinates, each making an angle of 60° with the preceding; we proceed similarly in finding the other coefficients. A separate diagram must be drawn for each pair of coefficients.

More generally, if we divide the complete period into n equal intervals of width $2\,\pi/n$ and measure the n ordinates, then (p. 177)

$$\frac{n}{2}\,a_k = \sum y_r \cos kx_r = y_0 \cos 0 + y_1 \cos k\left(\frac{2\pi}{n}\right) + \cdots + y_{n-1} \cos k\,\frac{2(n-1)\pi}{n},$$

$$\frac{n}{2}\,b_k = \sum y_r \sin kx_r = y_0 \sin 0 + y_1 \sin k\left(\frac{2\pi}{n}\right) + \cdots + y_{n-1} \sin k\,\frac{2(n-1)\pi}{n}.$$

Hence, if we construct the polygon of co-planar forces by starting at a point O and laying off in succession the ordinates, each making an angle $2\,k\pi/n$ with the preceding, then OA, the projection of the resultant OP on the horizontal, is equal to $na_k/2$, and OB, the projection of the resultant OP on the vertical, is equal to $nb_k/2$, except when $k = 0$ or $k = n/2$, when we get the values na_0, nb_0, $na_{n/2}$, $nb_{n/2}$, respectively. Furthermore, the length of OP is $n/2$ (or n) times the amplitude c_k and the angle between OP and OB gives the phase ϕ_k of the complete harmonic $c_k \sin(kx + \phi_k)$.

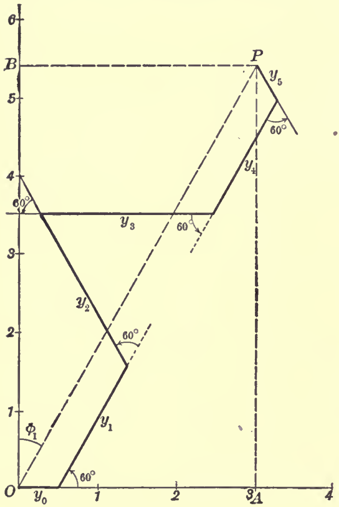


FIG. 93b.

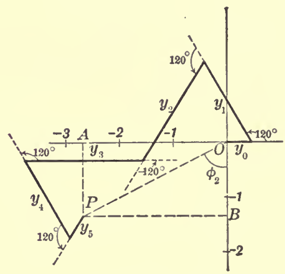


FIG. 93c.

Example. Analyze graphically the periodic curve in Fig. 86b.

As in the example on p. 181, we shall find the first three harmonics from the data

x	0°	60°	120°	180°	240°	300°
y	0.47	1.77	2.20	−2.20	−1.64	−0.49

Here

$$
\begin{aligned}
6\,a_0 &= y_0 + y_1 + \cdots + y_5 = 0.11; & a_0 &= 0.02.\\
6\,a_3 &= y_0 - y_1 + \cdots - y_5 = 1.95; & a_3 &= 0.33.\\
3\,a_1 &= OA \text{ (Fig. 93}b\text{)} = 3.09; & a_1 &= 1.03.\\
3\,b_1 &= OB \text{ (Fig. 93}b\text{)} = 5.35; & b_1 &= 1.78.\\
3\,c_1 &= OP \text{ (Fig. 93}b\text{)} = 6.25; & c_1 &= 2.08, & \phi_1 &= 30^\circ.\\
3\,a_2 &= OA \text{ (Fig. 93}c\text{)} = -2.67; & a_2 &= -0.89.\\
3\,b_2 &= OB \text{ (Fig. 93}c\text{)} = -1.35; & b_2 &= -0.45.\\
3\,c_2 &= OP \text{ (Fig. 93}c\text{)} = 3.00; & c_2 &= 1.00, & \phi_2 &= -60^\circ.
\end{aligned}
$$

Result:

$$y = 0.02 + 1.03 \cos x - 0.89 \cos 2x + 0.33 \cos 3x$$
$$+ 1.78 \sin x - 0.45 \sin 2x$$
$$= 0.02 + 2.08 \sin (x + 30°) + \sin (2x - 60°) - 0.33 \sin (3x - 90°).$$

Note the close agreement of this result with that obtained by the arithmetic method on p. 181.

94. Mechanical evaluation of the coefficients. Harmonic analyzers. — A very large number of machines have been constructed for finding the coefficients in Fourier's series by mechanical means. These instruments are called *harmonic analyzers*. The machines have done useful work where a large number of curves are to be analyzed. Among these analyzers we may mention that of Lord Kelvin,* Henrici,† Sharp,‡ Yule,§ Michelson and Stratton,|| Boucherot,¶ Mader,** and Westinghouse.†† We shall briefly describe the principles upon which the construction of two of these instruments depend.‡‡

The harmonic analyzer of Henrici. This is one of a number of machines which use an integrating wheel like that attached to a planimeter or integrator §§ to evaluate the integrals occurring in the general expressions for the coefficients

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} y\,dx, \quad a_k = \frac{1}{\pi}\int_0^{2\pi} y \cos kx\,dx, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} y \sin kx\,dx$$

given on p. 174.

If the curve in Fig. 94*a* represents a complete period of the curve to be analyzed, then evidently

$$\int_0^{2\pi} y\,dx = \text{area } OABCDBO;$$

so that, if the tracing point of a planimeter is allowed to follow the curve *OABCDBO*, the integrating wheel will give the reading $2\pi a_0$, from which a_0 may be computed.

* Proc. Roy. Soc., xxvii, 1878, p. 371; Kelvin and Tait's Natural Philosophy.

† Phil. Mag., xxxviii, 1894, p. 110.

‡ Phil. Mag., xxxviii, 1894, p. 121.

§ Phil. Mag., xxxix, 1895, p. 367; The Electrician, March 22, 1895.

|| Phil. Mag., xlv, 1898, p. 85.

¶ Morin, Les Appareils d'Intégration, 1913, p. 179.

** Elektrotech. Zeit., xxxvi, 1909; Phys. Zeit., xi, 1910, p. 354.

†† The Electric Journal, xi, 1914, p. 91.

‡‡ Brief descriptions of all but the last of these may be found in Modern Instruments and Methods of Calculation, a handbook of the Napier Tercentenary Celebration, 1914.

§§ For the principle of the planimeter and integrator, see pp. 246, 250.

Integrating by parts, we may write

$$a_k = \frac{1}{\pi}\int_0^{2\pi} y\cos kx\,dx = \left|\frac{1}{k\pi}y\sin kx\right|_0^{2\pi} - \frac{1}{k\pi}\int_0^{2\pi}\sin kx\,dy = -\frac{1}{k\pi}\int_0^{2\pi}\sin kx\,dy,$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} y\sin kx\,dx = \left|-\frac{1}{k\pi}y\cos kx\right|_0^{2\pi} + \frac{1}{k\pi}\int_0^{2\pi}\cos kx\,dy = \frac{1}{k\pi}\int_0^{2\pi}\cos kx\,dy.$$

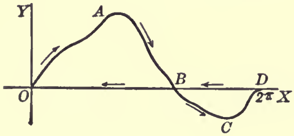


FIG. 94*a*.

Now if the planimeter carries two integrating wheels whose axes make at each instant angles kx and $\pi/2 - kx$ with the y-axis, and the point of intersection of these axes is capable of moving parallel to the y-axis, then as the tracer point passes around the boundary *OABCDBO*, these wheels give readings proportional to

$$\int \sin kx\,dy \quad \text{and} \quad \int \sin\left(\frac{\pi}{2} - kx\right)dy = \int \cos kx\,dy,$$

from which the values of a_k and b_k can be found.

In one form of the instrument the curve is drawn on a horizontal cylinder with the y-axis as one of the elements. A mechanism is attached to a carriage which moves along a rail parallel to the axis, by means of which a tracer point follows the curve while the cylinder rotates; the mechanism allows the axes of the integrating wheels to be turned through an angle kx while the cylinder rotates through an angle x. Coradi, the Swiss manufacturer, has perfected the instrument so that several pairs of coefficients may be read with a single tracing of the curve.

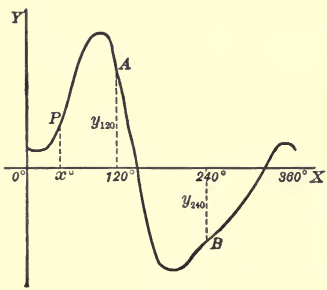


FIG. 94*b*.

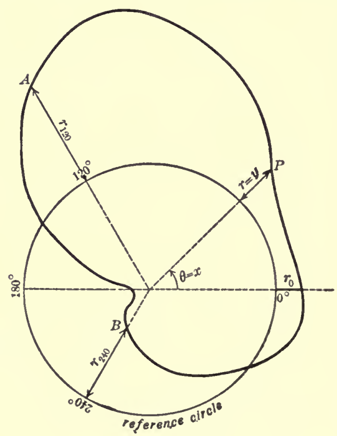


FIG. 94*c*.

The Westinghouse harmonic analyzer.—This machine, constructed by the Westinghouse Electric and Mfg. Co., is particularly useful in

analyzing the alternating voltage and current curves represented by a polar or circular oscillogram.

Fig. 94*b* gives one period of a periodic curve drawn on rectangular coördinate paper. In Fig. 94*c*, the same curve is represented on polar coördinate paper. This is done by constructing a circle of any convenient radius, called the zero line or reference circle and locating any point P by the angle $\theta = x$ and the radial distance $r = y$ from the zero line. Thus the points marked P, A, and B in Figs. 94*b* and 94*c* are corresponding points. If only the odd harmonics are present, the second half-period of the curve in Fig. 94*b* will be a repetition below the x-axis of the first half-period; in this case, the diameters at all angles of the curve in Fig. 94*c* will be equal, and equal to the diameter of the reference circle. The relation between r and θ,

$$\begin{aligned} r = f(\theta) &= a_1 \cos\theta + a_2 \cos 2\theta + \cdots + a_k \cos k\theta + \cdots \\ &\quad + b_1 \sin\theta + b_2 \sin 2\theta + \cdots + b_k \sin k\theta + \cdots, \end{aligned}$$

is the function to be analyzed. This is done as follows.

The circular record of the periodic curve, drawn by hand from the rectangular record or directly by the circular oscillograph,* is transferred

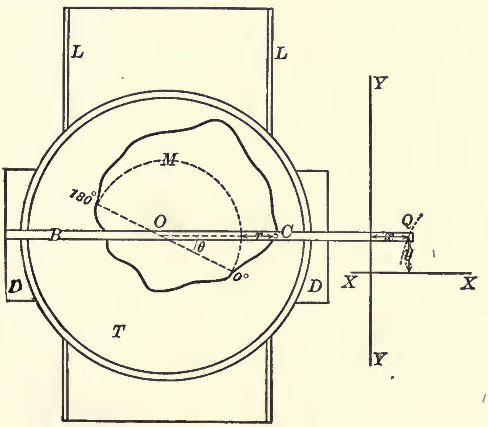


FIG. 94*d*.

to a card of bristol board and a template is prepared by cutting around the curve. In the initial position the template M (Fig. 94*d*) is secured on a turntable T so that the axis $\theta = 0$ lies under the transverse cross-bar B. The turntable is set on a carriage D which slides on the rails L. The

* The Electric Journal, xi, 1914, p. 262.

carriage is given an oscillatory motion by the motion of a crank-pin P (Figs. 94*e*, 94*f*) attached to a rotating gear G and sliding in a transverse slot S on the bottom of the carriage. The carriage thus has a simple harmonic motion whose amplitude is the crank-pin radius R. By means of a crank and a simple arrangement of gears, the carriage makes k complete oscillations while the template makes one revolution, when determining the kth harmonic.

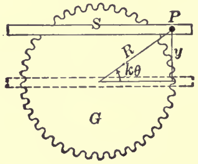


FIG. 94*e*.

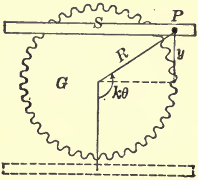


FIG. 94*f*.

The cross-bar B is attached to the oscillating carriage; this bar carries a pin C held in contact with the edge of the template by means of springs, so that the bar has a transverse motion as the template revolves. Referred to a pair of axes xx and yy, the motion of the end of the bar, $Q(x,y)$, may be said to consist of two components, viz., the transverse motion of the bar, $x = r = f(\theta)$, the function to be analyzed, and the simple harmonic motion of the carriage,

$$(1)\ y = R \sin k\theta \quad \text{or} \quad (2)\ y = R \sin\left(k\theta - \frac{\pi}{2}\right) = -R \cos k\theta,$$

according as the carriage is started with the slot S in the dotted position of Fig. 94*e* or of Fig. 94*f*. A planimeter is attached with its tracing point at Q. This point then describes compound Lissajous figures whose areas A_1 and A_2 may be read from the integrating wheel of the planimeter.

Now from (1), $\dfrac{dy}{d\theta} = Rk \cos k\theta$ and from (2) $\dfrac{dy}{d\theta} = Rk \sin k\theta$, hence

$$A_1 = \int_0^{2\pi} x\,dy = \int_0^{2\pi} rRk \cos k\theta\,d\theta = Rk \int_0^{2\pi} r \cos k\theta\,d\theta = Rk\pi a_k,$$

$$A_2 = \int_0^{2\pi} x\,dy = \int_0^{2\pi} rRk \sin k\theta\,d\theta = Rk \int_0^{2\pi} r \sin k\theta\,d\theta = Rk\pi b_k,$$

using the formulas for a_k and b_k on p. 174.

Therefore $$a_k = \frac{A_1}{Rk\pi}, \qquad b_k = \frac{A_2}{Rk\pi}.$$

Gears are provided to analyze for all even and odd harmonics from 1 to 50, and the shifting of the gears is a very simple matter.

EXERCISES.

1. Sketch the periodic curves

$$y = 2\cos x; \quad y = \cos 2x; \quad y = 2\cos x + \cos 2x.$$[1]

2. Sketch the periodic curves

$$y = 1 + \sin x; \quad y = -\tfrac{1}{2}\sin 2x; \quad y = \tfrac{1}{3}\sin 3x;$$
$$y = 1 + \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x.$$

3. Sketch the periodic curves

$$y = 2\sin(x - 40.5°); \quad y = \sin(2x + 72.3°); \quad y = \tfrac{1}{2}\sin(3x - 90°);$$
$$y = 2\sin(x - 40.5°) + \sin(2x + 72.3°) + \tfrac{1}{2}\sin(3x - 90°).$$

4. Sketch the periodic curve

$$y = \cos x + 0.4\cos 3x + 0.5\sin x - 0.5\sin 3x.$$

5. Sketch the periodic curve

$$y = \cos x + 0.4\cos 3x - 0.2\cos 5x + 0.5\sin x - 0.5\sin 3x - 0.3\sin 5x.$$

6. By use of the formulas on p. 177 and the direct method illustrated on p. 179, determine the coefficients of the third and fourth harmonics of the periodic curve in Fig. 89; use the table of ordinates on p. 179.

7. Determine the first three harmonics of the periodic curve given by the following data; use the computing form on p. 180.

x	0°	60°	120°	180°	240°	300°
y	−0.85	0.95	0.72	2.75	−1.37	−2.20

8. Determine the first six harmonics of the periodic curve given by the following data; use the computing form on p. 183.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
y	−18	−39	−39	−8	22	22	11	10	14	12	15	−1

9. Determine the first twelve harmonics of the periodic curve given by the following data; use the computing form on p. 185. (The curve is a graphical representation of the diurnal variation of the atmospheric electric potential gradient at Edinburgh during the year 1912.)

x	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°
y	−18	−30	−39	−41	−39	−32	−8	11	22	24	22	20
x	180°	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°
y	11	3	10	16	14	12	12	18	15	9	−1	−7

10. Devise computing forms for determining the even and odd harmonic coefficients using 8 and 16 ordinates respectively.

11. Determine the odd harmonics up to the fifth for the symmetric periodic curve given by the following data; use the computing form on p. 187.

x	0°	30°	60°	90°	120°	150°
y	0	676	660	940	1004	554

12. Determine the odd harmonics up to the fifth for the symmetric periodic curve from which the following measurements were taken; use the computing form on p. 187.

x	0°	30°	60°	90°	120°	150°
y	0	4	9.5	9	3	2

13. Determine the odd harmonics up to the eleventh for the symmetric periodic curve from which the following measurements were taken; use the computing form on p. 190.

x	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°
y	0	14	33	52	60	40	27	30	15	18	6	14

14. Determine the odd harmonics up to the seventeenth for the symmetric periodic curve from which the following measurements were taken; use the computing form on p. 192.

x	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°
y	0	5	9	21	20	21	27	30	29	33	42	44

120°	130°	140°	150°	160°	170°
38	46	45	30	31	29

15. Determine the first three harmonics for the periodic curve from which the following measurements were taken; use the method of selected ordinates in Art. 91; assume that all higher harmonics are absent.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°
y	10.0	5.0	5.3	7.2	6.0	−6.8	−10.9	−8.9	10.0	18.5	10.7	−3.4	−25.9

x	300	315	330
y	−17.3	−4.7	5.1

16 Determine the first three harmonics for the periodic curve drawn in Fig. 86*b*; use the method of selected ordinates in Art. 91.

17. Determine the first six harmonics for the periodic curve drawn in Fig. 89; use the method of selected ordinates in Art. 91; assume that all higher harmonics are absent.

18. Determine the first and third harmonics for the symmetric periodic curve given by the following data; use the method of selected ordinates in Art. 92; assume that all higher harmonics are absent

x	0°	30°	60°	90°	120°	150°
y	0	62.9	66.5	22.4	14.9	33.3

19. Assuming that the harmonics higher than the fifth are negligible, determine the odd harmonics of the symmetric periodic curve from which the following measurements were taken; use the method of selected ordinates in Art. 92.

x	0°	30°	60°	90°	120°	150°
y	0	676	660	940	1004	554

x	0°	18°	36°	54°	72°	90°	108°	126°	144°	162°
y	0	470	719	678	702	940	1086	920	639	375

20. Use the method of selected ordinates in Art. 92 to determine the ninth harmonic of the curve given by the table in Ex. 14.

21. Analyze graphically the curve in Ex. 7.

CHAPTER VIII.

INTERPOLATION.

95. Graphical Interpolation. — Having found the empirical formula connecting two measured quantities we may use this in the process of *interpolation*, *i.e.*, in computing the value of one of the quantities when the other is given within the range of values used in the determination of the formula. It is the purpose of this chapter to give some methods whereby interpolation may be performed when the empirical formula is inconvenient for computation or when such a formula cannot be found.

Let the following table represent a set of corresponding values of two quantities

x	x_0	x_1	x_2	x_3	...	x_n
y	y_0	y_1	y_2	y_3	...	y_n

where y is a known or an unknown function of x. Our problem is to find the value of $y = y_k$ for a value of $x = x_k$ between x_0 and x_n.

A simple graphical method consists in plotting the values of x and y as coördinates, drawing a smooth curve through or very near the plotted points, and measuring the ordinate y_k of the curve for the abscissa x_k. The value of y_k thus obtained may be sufficiently accurate for the purpose in hand. Thus from the curve in Fig. 72*b*, we read $t = 10$, $A = 77.0$, and $t = 30$, $A = 45.0$. If we use the empirical formula derived on p. 133,

$$A = 100.1\, e^{-0.0265\, t}, \quad \text{or} \quad \log A = 2.0005 - 0.0115\, t,$$

we compute $t = 10$, $A = 76.8$ and $t = 30$, $A = 45.2$. By comparison with the table on p. 132 we note that the measured values of A for $t = 10$ and $t = 30$ agree about as closely with the computed values as the neighboring observed values agree with their corresponding computed values. Here, the last significant figures in the values of A were used in constructing the plot.

On the other hand, in Fig. 71*c*, we read $v = 40$, $p = 10.00$, whereas the empirical formula on p. 131 gives $v = 40$, $p = 9.42$. The residual is 0.58, much larger than the residuals in the table on p. 130 for neighboring values of v. Here, the plot was constructed without using the last significant figures in the values of the quantities. It is of no advantage to construct a larger plot since the curve between plotted points is all the more indefinite.

For most problems the arithmetic or algebraic methods to be explained in the following sections give much bettei results.

96. Successive differences and the construction of tables. — Given a series of *equidistant* values of x and their corresponding values of y,

x	x_0	x_1	x_2	x_3	. . .	x_n
	x_0	$x_0 + h$	$x_0 + 2h$	$x_0 + 3h$	. . .	$x_0 + nh$
y	y_0	y_1	y_2	y_3	. . .	y_n,

we define the various orders of differences of y as follows:

1st difference $= \Delta^1$: $a_0 = y_1 - y_0,\ a_1 = y_2 - y_1,\ \ldots,\ a_{n-1} = y_n - y_{n-1}$;
2d difference $= \Delta^2$: $b_0 = a_1 - a_0,\ b_1 = a_2 - a_1,\ \ldots,\ b_{n-2} = a_{n-2} - a_{n-1}$;
3d difference $= \Delta^3$: $c_0 = b_1 - b_0,\ c_1 = b_2 - b_1,\ \ldots,\ c_{n-3} = b_{n-3} - b_{n-2}$;

. . . .

kth difference $= \Delta^k$: $k_0 = j_1 - j_0,\ k_1 = j_2 - j_1,\ \ldots$.

These may be tabulated as follows:

x	y	Δ^1	Δ^2	Δ^3	Δ^4	. . .	Δ^k . . .
$x_0 = x_0$	y_0						
		a_0					
$x_1 = x_0 + h$	y_1		b_0				
		a_1		c_0			
$x_2 = x_0 + 2h$	y_2		b_1		d_0		
		a_2		c_1	.		
$x_3 = x_0 + 3h$	y_3		b_2	.	.		
		a_3	.	.	.		k_0
$x_4 = x_4 + 4h$	y_4	.	.	.	.		
.	.	.	.	.	.		k_1
.	.	.	.	.	.		.
.	.	.	.	.	.		.
.	.	.	.	.	.		.
.	.	.	.	.	.		.
$x_{n-1} = x_0 + (n-1)h$	y_{n-1}						.
		a_{n-1}					.
$x_n = x_0 + nh$	y_n						

where a quantity in any column of differences is written between two quantities in the preceding column and is equal to the lower one of these minus the upper one.

We may apply the above definitions in the formation of the differences of y when $y = f(x)$; thus,

$\Delta y = f(x + h) - f(x) = \Delta f(x);\ \Delta^2 y = \Delta f(x + h) - \Delta f(x) = \Delta^2 f(x)$; etc. *E.g.*, if

$$y = x^2 - 2x + 2,\ \Delta y = [(x + h)^2 - 2(x + h) + 2] - [x^2 - 2x + 2]$$
$$= 2hx + (h^2 - 2h);$$
$$\Delta^2 y = [2h(x + h) + (h^2 - 2h)] - [2hx + (h^2 - 2h)]$$
$$= 2h^2.$$

We note that $\Delta^2 y = 2h^2$, so that the second differences are constant for all values of x.

Similarly, if $y = x^n$ where n is a positive integer,

$$\Delta y = (x + h)^n - x^n = x^n + nx^{n-1}h + \frac{n(n-1)}{\lfloor\underline{2}}x^{n-2}h^2 + \cdots - x^n$$

$$= nx^{n-1}h + \frac{n(n-1)}{\lfloor\underline{2}}x^{n-2}h^2 + \cdots + h^n;$$

$$\Delta^2 y = \left[n(x+h)^{n-1}h + \frac{n(n-1)}{\lfloor\underline{2}}(x+h)^{n-2}h^2 + \cdots\right] - \left[nx^{n-1}h + \frac{n(n-1)}{\lfloor\underline{2}}x^{n-2}h^2 + \cdots\right]$$

$$= n(n-1)x^{n-2}h^2 + \cdots,$$

$$\Delta^3 y = n(n-1)(n-2)x^{n-3}h^3 + \cdots,$$

$$\cdots\cdots\cdots\cdots$$

$$\Delta^n y = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1\, h^n = \lfloor\underline{n}\, h^n;$$

hence the nth differences of x^n, where n is a positive integer, are constant, and hence the nth differences of any polynomial of the nth degree

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Kx + L,$$

where n is a positive integer, are constant. If in forming the differences of a function some order of differences, say the nth, becomes approximately constant, then we may say that the function can be represented approximately by a polynomial of the nth degree, where n is a positive integer.

The formation of the differences for various functions is illustrated in the following tables:

(1) $y = x^3$

x	y	Δ^1	Δ^2	Δ^3
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		6
6	216		36	
		127		6
7	343		42	
		169		6
8	512		48	
		217		
9	729			

(2) $y = x^3$

x	y	Δ^1	Δ^2
5.16	137.39		
		4.03	
5.21	141.42		0.08
		4.11	
5.26	145.53		0.08
		4.19	
5.31	149.72		0.08
		4.27	
5.36	153.99		0.08
		4.35	
5.41	158.34		0.08
		4.43	
5.46	162.77		0.08
		4.51	
5.51	167.28		0.09
		4.60	
5.56	171.88		

(3) $y = \sqrt[3]{x}$

x	y	Δ^1	Δ^2
20	2.7144		
		445	
21	2.7589		−14
		431	
22	2.8020		−12
		419	
23	2.8439		−13
		406	
24	2.8845		−11
		395	
25	2.9240		

(4) $y = \sqrt[3]{x}$

x	y	Δ^1
611	8.4856	
		46
612	8.4902	
		46
613	8.4948	
		46
614	8.4994	
		46
615	8.5040	
		46
616	8.5086	

(5) *Train-resistance*

V speed in mi. per hr.	R resist. in lbs. per ton	Δ^1	Δ^2
20	5.5		
		3.6	
40	9.1		2.2
		5.8	
60	14.9		2.1
		7.9	
80	22.8		2.4
		10.5	
100	33.3		2.2
		12.7	
120	46.0		

(6) *Speed of a vessel*

V speed in knots per hr.	I horse-power	Δ^1	Δ^2	Δ^3
8	1,000			
		400		
9	1,400		100	
		500		0
10	1,900		100	
		600		50
11	2,500		150	
		750		50
12	3,250		200	
		950		50
13	4,200		250	
		1200		100
14	5,400		350	
		1550		100
15	6,950		450	
		2000		50
16	8,950		500	
		2500		
17	11,450			

(7) $y = \log x$

x	y	Δ^1
500	2.6990	
		8
501	2.6998	
		9
502	2.7007	
		9
503	2.7016	
		8
504	2.7024	
		9
505	2.7033	
		9
506	2.7042	

(8) $y = \log \sin x$

x	y	Δ^1	Δ^2	Δ^3
1° 0′	8.2419−10			
		669		
1° 10′	8.3088−10		−89	
		580		20
1° 20′	8.3668−10		−69	
		511		16
1° 30′	8.4179−10		−53	
		458		8
1° 40′	8.4637−10		−45	
		413		10
1° 50′	8.5050−10		−35	
		378		
2° 0′	8.5428−10			

In the above tables we note the following:

In (1), $y = x^3$ and Δ^3 is constant.

In (2), $y = x^3$ and Δ^2 is constant since we have carried the work to two decimal places and Δ^3 does not sensibly affect the second decimal place.

If the computation had been carried to six decimal places, Δ^2 would not be constant but Δ^3 would be.

In (3), Δ^2 is approximately constant, so that if we desire to work to four decimal places, $\sqrt[3]{x}$ could be represented by a polynomial of the second degree within the given range of values of x.

In (4), Δ^1 is approximately constant so that $\sqrt[3]{x}$ could be represented by an equivalent polynomial of the first degree.

In (5) and (6), Δ^2 and Δ^3 are approximately constant, so that R may be approximately represented by a polynomial of the second degree in V, and I by a polynomial of the third degree in V.

In (7), $\log x$ may be approximately represented by a polynomial of the first degree, and in (8), $\log \sin x$ by a polynomial of the third degree within the given range of values of x.

In general, it is evident that we may stop the process of finding successive differences much sooner the smaller the number of digits required and the smaller the constant interval h. We should stop immediately if the differences become irregular.

The formation of differences is often valuable where a function is to be tabulated for a set of values of the variable. Thus, suppose we wish to form a table for $y = \pi x^2/4$, expressing the area of a circle in terms of the diameter, for equidistant values of x. Since we have a polynomial of the second degree, $\Delta^2 y$ is constant, and if $h = 1$ and the work is to be carried to 4 decimal places, we need merely compute the values of y for $x = 1, 2, 3$ and form the corresponding differences; proceeding backwards, we repeat the value of $\Delta^2 y = 1.5708$, add this to $\Delta y = 3.9270$ and get 5.4978, add this to 7.0686 and get 12.5664, which is the value of y for $x = 4$. We proceed in the same manner to get the values of y for successive values of x.

x	$y = \pi x^2/4$	Δ^1	Δ^2
1	0.7854		
		2.3562	
2	3.1416		1.5708
		3.9270	
3	7.0086		1.5708
		5.4978	
4	12.5664		1.5708
		7.0686	
5	19.6350		

x	$y = \pi x^2/4$	Δ^1	Δ^2
69	3739.28		
		109.17	
70	3848.45		1.57
		110.74	
71	3959.19		1.57
		112.31	
72	4071.50		1.57
		113.88	
73	4185.38		

For larger values of x where we wish to work to two decimal places only, we take $\Delta^2 y = 1.57$ and proceed as above.

Suppose we wish to tabulate the function $y = x^3$. Here Δ^3 is constant so that we merely compute the part of the accompanying table in heavy type. Then we extend the column for Δ^3 by inserting 6's, extend the columns for Δ^2 and Δ^1 by simple additions and subtractions, and thus determine the values of x^3 for all integral values of x.

x	$y = x^3$	Δ^1	Δ^2	Δ^3
−1	−1			
		1		
0	0		0	
		1		6
1	1		6	
		7		6
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		
6	216			

The same procedure may be followed in the construction of a table for a function where a certain order of differences is only approximately constant. Thus, in forming table (4) of cube roots, we note that for that portion of the table Δy is approximately 0.0046 so that we can find the values of $\sqrt[3]{x}$ by simple additions; we must check the work by direct computation every few values in order to find when $\Delta^2 y$ changes its value.

97. Newton's interpolation formula. — We shall now express the value of y for any value of x. From the definitions of successive differences we have

$$y_1 = y_0 + a_0; \quad y_2 = y_1 + a_1 = (y_0 + a_0) + (a_0 + b_0) = y_0 + 2\,a_0 + b_0;$$
$$y_3 = y_2 + a_2 = (y_0 + 2\,a_0 + b_0) + (a_0 + 2\,b_0 + c_0) = y_0 + 3\,a_0 + 3\,b_0 + c_0;$$
$$y_4 = y_3 + a_3 = (y_0 + 3\,a_0 + 3\,b_0 + c_0) + (a_0 + 3\,b_0 + 3\,c_0 + d_0)$$
$$= y_0 + 4\,a_0 + 6\,b_0 + 4\,c_0 + d_0;$$

. . . .

We note that the coefficients are those of the binomial expansion, and this suggests that

$$y_n = y_0 + na_0 + \frac{n\,(n-1)}{\underline{|2}}\,b_0 + \frac{n\,(n-1)\,(n-2)}{\underline{|3}}\,c_0 + \cdots, \tag{I}$$

where n is a positive integer. If this equation is true, then, replacing y by a, the first difference, we may also write

$$a_n = a_0 + nb_0 + \frac{n\,(n-1)}{\underline{|2}}\,c_0 + \frac{n\,(n-1)\,(n-2)}{\underline{|3}}\,d_0 + \cdots,$$

$$\therefore \quad y_{n+1} = y_n + a_n = y_0 + (n+1)\,a_0 + \left[\frac{n\,(n-1)}{\underline{|2}} + n\right] b_0$$
$$+ \left[\frac{n\,(n-1)\,(n-2)}{\underline{|3}} + \frac{n\,(n-1)}{\underline{|2}}\right] c_0 + \cdots$$
$$= y_0 + (n+1)\,a_0 + \frac{(n+1)\,n}{\underline{|2}}\,b_0 + \frac{(n+1)\,n\,(n-1)}{\underline{|3}}\,c_0 + \cdots,$$

where the coefficients are again those of the binomial expansion with n replaced by $n + 1$. Thus we have shown that if equation (I) is true for any positive integral value of n, it is true for the next larger integral value. But we have shown (I) to be true when $n = 4$, therefore it is true when $n = 5$; since it is true for $n = 5$, therefore it is true for $n = 6$; etc. Hence (I) is true for all positive integral values of n.

Now if some order of differences, say the kth order, is constant, *i.e.*, $\Delta^k y = k_0$, then y is a polynomial of the kth degree in n, and equation (I) may be written

$$A + Bn + Cn^2 + \cdots + Ln^k = y_0 + na_0 + \frac{n(n-1)}{\lfloor 2}b_0 + \ldots + \frac{n(n-1)\ldots(n-k+1)}{\lfloor k}k_0.$$

The right member of this equation is also a polynomial of the kth degree in n, and since these polynomials are equal for *all* positive integral values of n (*i.e.*, for more than k values of n), they must be equal for all values of n, integral, fractional, positive, and negative.

Hence if the kth order of differences is constant, we have

$$y_n = y_0 + na_0 + \frac{n(n-1)}{\lfloor 2}b_0 + \frac{n(n-1)(n-2)}{\lfloor 3}c_0 + \cdots + \frac{n(n-1)\ldots(n-k+1)}{\lfloor k}k_0 \quad \text{(N)}$$

for *all* values of n. This fundamental formula of interpolation is known as *Newton's interpolation formula*. In this formula, y_0 is any one of the tabulated values of y and the differences are those which occur in a line through y_0 and parallel to the upper side of the triangle in the tabular scheme on p. 210.

Newton's formula is approximately true for the more frequent case where the differences of some order are approximately constant; all the more so if $n < 1$. We can always arrange to have $n < 1$; for if we wish to find the value of $y = Y$ for $x = X$, where X lies between the tabular values x_i and x_{i+1}, we use Newton's formula with y_i and the corresponding differences $a_i, b_i, c_i, \ldots$, so that $X = x_i + nh$ and $n = \dfrac{X - x_i}{h} < 1$.* The values of the binomial coefficients occurring in the formula have been tabulated for values of n between 0 and 1 at intervals of 0.01.†

Let us now apply Newton's formula to the illustrative difference-tables (1) to (8).

* The ordinary interpolation formula of proportional parts disregards all differences higher than the first, so that $y = y_0 + na_0$, where $n = (X - x_0)/h$. This simple formula will often give the desired degree of accuracy if the interval h can be made small enough.

† See H. L. Rice, *Theory and Method of Interpolation*.

(1) To compute $(3.4)^3$; $y_0=27$, $h=1$, $n=(3.4-3)/1=0.4$;

$$\therefore \quad (3.4)^3 = 27+(0.4)(37)+\frac{(0.4)(-0.6)}{2}(24)+\frac{(0.4)(-0.6)(-1.6)}{6}(6)$$
$$= 39.304.$$

(3) To compute $\sqrt[3]{23.5}$; $y_0 = 2.8439$, $h = 1$, $n = (23.5 - 23)/1 = 0.5$;

$\therefore \quad \sqrt[3]{23.5} = 2.8439 + \frac{1}{2}(0.0406) + \frac{1}{8}(0.0011) = 2.8643.$

If we use the ordinary interpolation formula of proportional parts, $\sqrt[3]{23.5} = 2.8439 + \frac{1}{2}(0.0406) = 2.8642$, which would be correct to three decimals only.

(4) To compute $\sqrt[3]{612.25}$; $y_0=8.4902$, $h=1$, $n=(612.25-612)/1=\frac{1}{4}$;

$\therefore \quad \sqrt[3]{612.25} = 8.4902 + \frac{1}{4}(0.0046) = 8.4914.$

(5) To compute R when $V=65$; $R_0=14.9$, $h=20$, $n=(65-60)/20=\frac{1}{4}$;

$\therefore \quad R = 14.9 + \frac{1}{4}(7.9) - \frac{3}{32}(2.4) = 16.7.$

(7) To compute log 501.3; $y_0=2.6998$, $h=1$, $n=(501.3-501)/1=0.3$;

$\therefore \quad \log 501.3 = 2.6998 + 0.3\,(0.0009) = 2.7001.$

(8) To compute log sin 1° 16′; $y_0 = 8.3088 - 10$, $h = 10'$, $n = (1°\,16' - 1°\,10')/10' = 0.6$;

$$\therefore \quad \log\sin 1°\,16' = (8.3088 - 10) + 0.6\,(0.0580) - 0.12\,(-0.0069)$$
$$+ 0.056\,(0.0016) = 8.3445 - 10, \text{ correct to 4 decimals.}$$

If we use the ordinary formula of proportional parts, we have log sin 1° 16′ = 8.3088 − 10 + 0.6 (0.0580) = 8.3436 − 10, correct to 2 decimals only.

If the value of x for which we wish to determine the value of y is near the end of the table we may not have all the required differences. To take care of this case Newton's formula is slightly modified. If we invert the series of values of x in the tabular scheme on p. 210, and form the differences, we have

x	y	Δ^1	Δ^2	Δ^3	Δ^4
x_n	y_n				
		$-a_{n-1}$			
x_{n-1}	y_{n-1}				
.					
.					
.					
.					
.					
x_4	y_4				
		$-a_3$			
x_3	y_3		b_2		
		$-a_2$		$-c_1$	
x_2	y_2		b_1		d_0
		$-a_1$		$-c_0$	
x_1	y_1		b_0		
		$-a_0$			
x_0	y_0				

Starting at y_4 and applying Newton's formula, we get

$$y_n = y_4 + n(-a_3) + \frac{n(n-1)}{\underline{|2}} b_2 + \frac{n(n-1)(n-2)}{\underline{|3}}(-c_1) + \cdots$$

$$= y_4 - na_3 + \frac{n(n-1)}{\underline{|2}} b_2 - \frac{n(n-1)(n-2)}{\underline{|3}} c_1 + \cdots.$$

Comparing the result with the scheme on p. 210, we note that the differences are those which occur along a line parallel to the lower side of the triangle in that scheme. Here y_4 is any value of y, and if X lies between x_4 and x_3, then $X = x_4 - nh$, and $n = (x_4 - X)/h$.

Example. To compute $\sqrt[3]{24.8}$. In table (3), $y_4 = 2.9240$, $h = 1$, $n = (25 - 24.8)/1 = 0.2$;

$$\therefore \sqrt[3]{24.8} = 2.9240 - 0.2(0.0395) + \frac{0.2(-0.8)}{2}(-0.0011) = 2.9162.$$

If a series of corresponding numerical values of two quantities are given, we may use Newton's formula for finding the polynomial which will represent this series of values exactly or approximately. For this purpose we replace n by $(x - x_0)/h$.

Thus, in table (1), $h = 1$, $x_0 = 1$, $n = x - 1$;

$$\therefore \quad y = 1 + (x-1)7 + \frac{(x-1)(x-2)}{\underline{|2}} 12 + \frac{x(x-1)(x-2)(x-3)}{\underline{|3}} 6 = x^3.$$

In table (5), $h = 20$, $V_0 = 20$, $n = \dfrac{V-20}{20} = \dfrac{V}{20} - 1$;

$$\therefore \quad R = 5.5 + \left(\frac{V}{20} - 1\right)3.6 + \frac{\left(\frac{V}{20} - 1\right)\left(\frac{V}{20} - 2\right)}{\underline{|2}} 2.2$$

$$= 4.1 + 0.015\,V + 0.00275\,V^2.$$

The values of R computed by this formula agree quite closely with those in the table.

In table (6), $h = 1$, $V_0 = 10$, $n = V - 10$;

$$\therefore \quad I = 1900 + (V-10)600 + \frac{(V-10)(V-11)}{2} 150$$

$$+ \frac{(V-10)(V-11)(V-12)}{6} 50$$

$$= -6850 + 2042\,V - 200\,V^2 + 8\tfrac{1}{3}\,V^3.$$

The values of I computed by this formula agree quite closely with those in the table; thus, $V = 12$ gives $I = 3254$.

Various formulas of interpolation similar to Newton's have been derived which are very convenient in certain problems. Among these may be mentioned the formulas of Stirling, Gauss, and Bessel.*

* For an account of these formulas, see H. L. Rice, *Theory and Practice of Interpolation*, and D. Gibb, *Interpolation and Numerical Integration.*

98. Lagrange's formula of interpolation. — Newton's formula is applicable only when the values of x are equidistant. When this is not the case, we may use a formula known as Lagrange's formula. Given the following table of values of x and y,

x	a_1	a_2	a_3	a_4	a_5	$\dots$	a_n
y	y_1	y_2	y_3	y_4	y_5	$\dots$	y_n

we are to find an expression for y corresponding to a value of x lying between a_1 and a_n. We take for y an expression of the $(n-1)$st degree in x containing n constants, and determine these n constants by requiring the n sets of values of x and y to satisfy the equation. But instead of assuming the form $y = A + Bx + Cx^2 + \cdots + Nx^{n-1}$, we may assume the equivalent form

$$\begin{aligned} y = & A\,(x-a_2)(x-a_3)(x-a_4)\,\dots\,(x-a_n) \\ & + B\,(x-a_1)(x-a_3)(x-a_4)\,\dots\,(x-a_n) \\ & + C\,(x-a_1)(x-a_2)(x-a_4)\,\dots\,(x-a_n) \\ & + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + N\,(x-a_1)(x-a_2)(x-a_3)\,\dots\,(x-a_{n-1}), \end{aligned}$$

where the n terms in the right member of the equation lack the factors $(x-a_1)$, $(x-a_2)$, $\dots$ $(x-a_n)$ respectively.

Since (a_1, y_1) is to satisfy this equation,

$$y_1 = A\,(a_1-a_2)(a_1-a_3)(a_1-a_4)\,\dots\,(a_1-a_n),$$

since all the other terms contain the factor $(a_1 - a_1)$ and therefore vanish.

Similarly,

$$\begin{aligned} y_2 &= B\,(a_2-a_1)(a_2-a_3)(a_2-a_4)\,\dots\,(a_2-a_n), \\ y_3 &= C\,(a_3-a_1)(a_3-a_2)(a_3-a_4)\,\dots\,(a_3-a_n), \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_n &= N\,(a_n-a_1)(a_n-a_2)(a_n-a_3)\,\dots\,(a_n-a_{n-1}). \end{aligned}$$

Hence,

$$A = \frac{y_1}{(a_1-a_2)(a_1-a_3)(a_1-a_4)\,\dots\,(a_1-a_n)},$$

$$B = \frac{y_2}{(a_2-a_1)(a_2-a_3)(a_2-a_4)\,\dots\,(a_2-a_n)}, \text{ etc.,}$$

and, finally,

$$\begin{aligned} y = y_1 & \frac{(x-a_2)(x-a_3)\,\dots\,(x-a_n)}{(a_1-a_2)(a_1-a_3)\,\dots\,(a_1-a_n)} + y_2 \frac{(x-a_1)(x-a_3)\,\dots\,(x-a_n)}{(a_2-a_1)(a_2-a_3)\,\dots\,(a_2-a_n)} \\ & + \cdots + y_n \frac{(x-a_1)(x-a_2)\,\dots\,(x-a_{n-1})}{(a_n-a_1)(a_n-a_2)\,\dots\,(a_n-a_{n-1})}. \end{aligned}$$

We note that in the term containing y_k, the numerator of the fraction lacks the factor $(x - a_k)$ and the denominator lacks the corresponding factor $(a_k - a_k)$. Lagrange's formula is in convenient form for logarithmic computation.

Example. In the table on p. 132 we have

t	14	17	31	35
A	68.7	64.0	44.0	39.1

and we are to find the value of A when $t=27$. Using Lagrange's formula,

$$A = 68.7\frac{(27-17)(27-31)(27-35)}{(14-17)(14-31)(14-35)} + 64.0\frac{(27-14)(27-31)(27-35)}{(17-14)(17-31)(17-35)}$$
$$+44.0\frac{(27-14)(27-17)(27-35)}{(31-14)(31-17)(31-35)} + 39.1\frac{(27-14)(27-17)(27-31)}{(35-14)(35-17)(35-31)}$$
$$= -20.5 + 35.2 + 48.0 - 13.4 = 49.3,$$

which agrees exactly with the observed value.

Example. In the table on p. 157 we have

t	0.1	0.2	0.4	0.8
i	2.48	2.66	2.58	2.00

and we are to find the value of i when $t = 0.3$. Using only the values $t = 0.2$ and $t = 0.4$,

$$i = 2.66\frac{(0.3-0.4)}{(0.2-0.4)} + 2.58\frac{(0.3-0.2)}{(0.4-0.2)} = 1.33 + 1.29 = 2.62.$$

Using all four values of t, $i = 2.68$. Using the empirical equation

$i = 4.94\, e^{-1.07t} - 2.85\, e^{-3.76t}$ (on p. 159), we get $i = 2.66$.

Gauss's interpolation formula for periodic functions. — When the data are periodic we may find the empirical equation as a trigonometric series by the method of Chapter VII and use this equation for purposes of interpolation, or we may use an equivalent equation given by Gauss:

$$y = y_1\frac{\sin\frac{1}{2}(x-a_2)\sin\frac{1}{2}(x-a_3)\ \ldots\ \sin\frac{1}{2}(x-a_n)}{\sin\frac{1}{2}(a_1-a_2)\sin\frac{1}{2}(a_1-a_3)\ \ldots\ \sin\frac{1}{2}(a_1-a_n)}$$
$$+ y_2\frac{\sin\frac{1}{2}(x-a_1)\sin\frac{1}{2}(x-a_3)\ \ldots\ \sin\frac{1}{2}(x-a_n)}{\sin\frac{1}{2}(a_2-a_1)\sin\frac{1}{2}(a_2-a_3)\ \ldots\ \sin\frac{1}{2}(a_2-a_n)}$$
$$+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

It is evident that $y = y_1$ when $x = a_1$, $y = y_2$ when $x = a_2$, etc., so that the equation is satisfied by the corresponding values of x and y.

99. Inverse interpolation. — Given the table

x	x_0	x_1	x_2	x_3	. . .	x_n
y	y_0	y_1	y_2	y_3	. . .	y_n

we may wish to find the value of x corresponding to a given value of y. If the values of x are equidistant we may use Newton's interpolation formula. Here we know y_n, y_0, a_0, b_0, c_0, . . . , and substituting these values in the formula we have an equation which is to be solved for n.

If only the first order of differences are taken into account, then $y_n = y_0 + na_0$, and $n = \dfrac{y_n - y_0}{a_0}$, the ordinary formula for inverse interpolation by proportional parts.

Example. In table (7), given $\log x = 2.7003$, to find x.

$$n = \frac{2.7003 - 2.6998}{0.0009} = 0.56, \text{ and } x = x_0 + nh = 501 + 0.56\,(1) = 501.56.$$

If only the first and second differences are taken into account, then $y_n = y_0 + na_0 + \dfrac{n\,(n-1)}{2} b_0$, a quadratic equation which can easily be solved for n.

Example. In table (5), given $R = 27.3$, to find V.

Here
$$27.3 = 22.8 + n\,(10.5) + \frac{n\,(n-1)}{2}(2.2),$$
or
$$1.1\, n^2 + 9.4\, n - 4.5 = 0;$$
hence $n = \frac{5}{11} = 0.455$ and $x = V_0 + nh = 80 + (0.455)\, 20 = 89.1$.

The empirical formula $R = 4.62 - 0.004\, V + 0.0029\, V^2$ on p. 149 gives
$$V = 89.1, \quad R = 27.3.$$

But if the third and higher orders of differences have to be taken into account, the method would require the solution of equations of the third and higher degrees. In such cases as well as in the case where the values of x are not equidistant, we may use Lagrange's formula and merely interchange x and y; *i.e.*,

$$x = x_1 \frac{(y - a_2)\,(y - a_3)\ \ldots}{(a_1 - a_2)\,(a_1 - a_3)\ \ldots} + x_2 \frac{(y - a_1)\,(y - a_3)\ \ldots}{(a_2 - a_1)\,(a_2 - a_3)\ \ldots} + \cdots.$$

Example. In table (8), given $\log \sin x = 8.3850 - 10$, to find x. Using only the following values,

$\log \sin x$	$8.3088 - 10$	$8.3668 - 10$	$8.4179 - 10$
x	$70'$	$80'$	$90'$

we have

$$\begin{aligned} x &= 70' \frac{(0.0182)\,(-0.0329)}{(-0.0580)\,(-0.1091)} + 80' \frac{(0.0762)\,(-0.0329)}{(0.0580)\,(-0.0511)} \\ &\quad + 90' \frac{(0.0762)\,(0.0182)}{(0.1091)\,(0.0511)} \\ &= 70'\,(-0.0946) + 80'\,(0.846) + 90'\,(0.249) \\ &= 83.47' = 1^\circ\, 23.47'. \end{aligned}$$

We may also use a method of *successive approximations* as follows: From Newton's formula we write

$$n = \frac{y - y_0}{a_0 + \frac{1}{2}(n-1)\, b_0 + \frac{1}{6}(n-1)\,(n-2)\, c_0 + \cdots}.$$

Applying this to the above example, and taking only the first differences into account, we get as a first approximation,

$$n_1 = \frac{y - y_0}{a_0} = \frac{(8.3850 - 10) - (8.3668 - 10)}{0.0511} = \frac{182}{511} = 0.356.$$

Taking also second differences into account and introducing the value of n_1 for n in the denominator, we get as a second approximation,

$$n_2 = \frac{y - y_0}{a_0 + \frac{1}{2}(n_1 - 1)b_0} = \frac{0.0182}{0.0511 + 0.0017} = \frac{182}{528} = 0.345.$$

We may continue in this way approximating more and more closely to the value of n. In this example it will be unnecessary to carry the work to third differences since Δ^3 is negligible. Hence

$$n = 0.345, \quad \text{and} \quad x = x_0 + nh = 1^\circ 20' + (0.345)(10') = 1^\circ 23.45'.$$

We may check this by direct interpolation. Here

$$y_0 = 8.3668 - 10, \quad h = 10', \text{ and } \quad n = 0.345;$$

hence,

$$y = 8.3668 - 10 + 0.345\,(0.0511) - 0.113\,(-0.0053) = 8.3850 - 10.$$

Example. Find the real root of the equation $x^3 + 5x - 1 = 0$. We form a table of differences of the function $y = x^3 + 5x - 1$.

x	y	Δ^1	Δ^2	Δ^3
−2	−19			
		12		
−1	−7		−6	
		6		6
0	−1		0	
		6		6
1	5		6	
		12		6
2	17		12	
		24		
3	41			

The root lies between $x = 0$ and $x = 1$, and we are to find the value of x when $y = 0$. Using the method of successive approximations we have

$$n_1 = \frac{y - y_0}{a_0} = \frac{0 + 1}{6} = \frac{1}{6} = 0.1667,$$

$$n_2 = \frac{y - y_0}{a_0 + \frac{1}{2}(n_1 - 1)b_0} = \frac{1}{6 + \frac{1}{2}(\frac{1}{6} - 1)6} = \frac{2}{7} = 0.2857,$$

$$n_3 = \frac{y - y_0}{a_0 + \frac{1}{2}(n_2 - 1)b_0 + \frac{1}{6}(n_2 - 1)(n_2 - 2)c_0} = \frac{1}{6 - \frac{15}{7} + \frac{60}{49}} = \frac{49}{249} = 0.1968,$$

$$n_4 = \frac{y - y_0}{a_0 + \frac{1}{2}(n_3 - 1)b_0 + \frac{1}{6}(n_3 - 1)(n_3 - 2)c_0} = \frac{1}{6 - 2.4096 + 1.4483}$$

$$= \frac{1}{5.0387} = 0.1985.$$

Hence, $$x = x_0 + nh = 0.1985.$$

From the table

x	0.1985	0.19845	0.1984
y	0.00032	0.00006	−0.00019

we note that $x = 0.1984$ is the root correct to 4 decimals.

EXERCISES

1. Tabulate the values and differences of the following functions; h is the common interval.

(a) x^2, from $x = 5$ to $x = 12$ when $h = 1$; and from $x = 3$ to $x = 3.1$ when $h = 0.01$.

(b) $\sqrt{x}$, from $x = 1$ to $x = 10$, when $h = 1$, and from $x = 563$ to $x = 570$ when $h = 1$.

(c) $\frac{1}{x}$, from $x = 60$ to $x = 70$ when $h = 1$, and from $x = 260$ to $x = 262$ when $h = 0.2$.

(d) $\frac{\pi D^3}{6}$ (volume of a sphere), from $D = 1$ to $D = 1.8$ when $h = 0.1$.

(e) $\log x$, to 4 decimals, from $x = 356$ to $x = 362$ when $h = 1$.

(f) $\tan x$, to 4 decimals, from $x = 32°$ to $x = 33°$ when $h = 10'$.

(g) $\log \cos x$, to 4 decimals, from $x = 88° 10'$ to $x = 89° 20'$ when $h = 10'$.

(h) e^x, to 4 decimals, from $x = 0.8$ to $x = 0.9$ when $h = 0.01$.

(i) $\frac{1}{2}(\alpha - \sin\alpha)$, to 4 decimals (area of a segment of a circle subtending a central angle α, in radians) from $\alpha = 25°$ to $\alpha = 32°$ when $h = 1°$.

2. Tabulate the differences for the following experimental results and indicate for each case the degree of the polynomial that would best express the relation between the variables.

(a) S = stress in lbs. per sq. in. in steel wire used for winding guns, E = elongation in inches per inch.

S	10,000	20,000	30,000	40,000	50,000	60,000	70,000	80,000
E	0.00019	0.00057	0.00094	0.00134	0.00173	0.00216	0.00256	0.00297

(b) Q = cu. ft. of water per sec. flowing over a Thomson gauge notch; H = ft. of head.

H	1.2	1.4	1.6	1.8	2.0
Q	4.2	6.1	8.5	11.5	14.9

(c) P/a = load in lbs. per sq. in. which causes the failure of long wrought-iron columns with round ends, l/r = ratio of length of column to least radius of gyration of its cross-section.

l/r	140	180	220	260	300	340	380	420
P/a	12,800	7500	5000	3800	2800	2100	1700	1300

(d) e = volts, p = kilowatts in a core-loss curve for an electric motor.

e	40	60	80	100	120	140	160
p	0.63	1.36	2.18	3.00	3.93	6.22	8.59

(e) A = amplitude of vibration in inches of a long pendulum, t = time in min. since it was set swinging.

t	0	1	2	3	4	5	6
A	10	4.97	2.47	1.22	0.61	0.30	0.14

(f) V = potential difference in volts, A = current in amperes in an electric circuit.

A	2.97	3.97	4.97	5.97	6.97	7.97
V	65.0	61.0	58.25	56.25	55.1	54.3

(g)

x	1	3	5	7	9	11	13
y	6.42	8.50	11.03	14.03	17.53	21.55	26.12

(*h*)

x	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7
y	3.00	1.89	1.27	0.88	0.63	0.46	0.33	0.25	0.18	0.05

3. By the method of differences explained in Art. 96, extend the tabulation of the functions in Exs. 1 a, b, d, e, h, i, for several values of the variables beyond the range of values for which the tables were constructed.

4. Apply Newton's interpolation formula to the tables in Ex. 1.

(a) In Ex. 1 a, find x^2 when $x = 7.3$ and $x = 3.056$.

(b) In Ex. 1 b, find $\sqrt{x}$ when $x = 566.2$.

(c) In Ex. 1 d, find $\pi D^3/6$ when $D = 1.452$.

(d) In Ex. 1 e, find $\log x$ when $x = 361.4$.

(e) In Ex. 1 g, find $\log \cos x$ when $x = 88° 43'$.

5. Apply Newton's interpolation formula to the tables in Ex. 2.

(a) In Ex. 2 a, find E when $S = 42{,}000$.

(b) In Ex. 2 b, find Q when $H = 1.7$, and compare with the value given by the empirical formula $Q = 2.672\, H^{2.48}$.

(c) In Ex. 2 c, find P/a when $l/r = 327$, and compare with the value given by the empirical formula $P/a = 417{,}000{,}000\, (l/r)^{2.1}$

(d) In Ex. 2 f, find V when $A = 4.07$.

(e) In Ex. 2 g, find y when $x = 6$.

(f) In Ex. 2 h, find y when $x = 1.3$ and $x = 2.46$.

6. In the following table (taken from p. 129)

θ	288	293	313	333
S	35.2	37.2	45.8	55.2

S is the number of grams of anhydrous ammonium chloride which dissolved in 100 grams of water makes a saturated solution of $\theta°$ absolute temperature. Use Lagrange's formula of interpolation to find S when $\theta = 300°$, using (1) only two values of θ, (2) three values of θ, (3) all four values of θ. Compare the results with the value given by the empirical formula $S = 0.000000882\, \theta^{3.09}$.

7. In the following table (taken from p. 141)

i	1	2	4	8
V	120	94	75	62

i is the current and V is the voltage consumed by a magnetite arc. Use Lagrange's formula to find V when $i = 3$, and compare the result with the value given by the empirical formula $V = 30.4 + 90.4\, i^{-0.507}$.

8. Use the methods of inverse interpolation (Art. 99) in the following:

(a) In Ex. 1 a, find x when $x^2 = 39$ and when $x^2 = 9.34$.

(b) In Ex. 1 e, find x when $\log x = 2.5542$.

(c) In Ex. 1 g, find x when $\log \cos x = 8.3946 - 10$.

(d) In Ex. 2 a, find S when $E = 0.00192$.

(e) In Ex. 2 c, find l/r when $P/a = 4000$.

(f) In Ex. 2 g, find x when $y = 15.25$.

9. Approximate to the real roots of the equations:

(a) $x^3 - 2x + 3 = 0$.

(b) $x^4 - 4x + 2 = 0$.

(c) $e^x + x^2 - 4 = 0$.

(d) $10 \log x - x - 2 = 0$.

(e) $\sin x + x^2 - 1.5 = 0$.

CHAPTER IX.

APPROXIMATE INTEGRATION AND DIFFERENTIATION.

100. The necessity for approximate methods. — In a large number of engineering problems it is necessary to determine the value of the definite integral, $\int_a^b f(x)\,dx$. Geometrically, this integral represents the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$. Physically, it may represent the work done by an engine, the velocity acquired by a moving body, the pressure on an immersed surface, etc. If $f(x)$ is analytically known, the above integral may be evaluated by the methods of the Integral Calculus. But if we merely know a set of values of $f(x)$ for various values of x, or if the curve is drawn mechanically, *e.g.*, an indicator diagram or oscillograph, or even where the function is analytically known but the integration cannot be performed by the elementary methods of the Integral Calculus — in all these cases, the integral must be evaluated by approximate methods — numerical, graphical, or mechanical. The planimeter is ordinarily used in measuring the area enclosed by an indicator diagram and in certain problems in Naval Architecture; such approximations often have the desired degree of accuracy. Where a higher degree of accuracy is required or where a planimeter is not available numerical methods must be used.

In certain problems it becomes necessary to determine the value of the derivative, $\frac{dy}{dx}$. Geometrically, this represents the slope of the curve $y = f(x)$ at any point. Physically, it arises in problems in which the velocity and acceleration are to be found when the distance is given as a function of the time, in problems involving maximum and minimum values and rates of change of various physical quantities, etc. To evaluate the derivative we may use the methods of the Differential Calculus if the function is analytically known. Otherwise we are forced to use approximate methods — numerical, graphical, or mechanical.

It is our purpose, in the following sections, to develop some of the numerical, graphical, and mechanical methods used in approximate integration and differentiation.

101. Rectangular, Trapezoidal, Simpson's, and Durand's rules. — Suppose we wish to find the approximate area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = x_0$ and $x = x_n$ (Fig. 101).

We divide the interval from $x = x_0$ to $x = x_n$ into n equal intervals of width h, and measure the $(n + 1)$ ordinates $y_0, y_1, y_2, \ldots, y_{n-1}, y_n$.

(1) *Rectangular rule.* — If, starting at P_0, we draw segments parallel to the x-axis through the points $P_0, P_1, P_2, \ldots, P_{n-1}$, the area enclosed by the rectangles thus formed is given by

$$A_R = h\,(y_0 + y_1 + y_2 + \cdots + y_{n-1}).$$

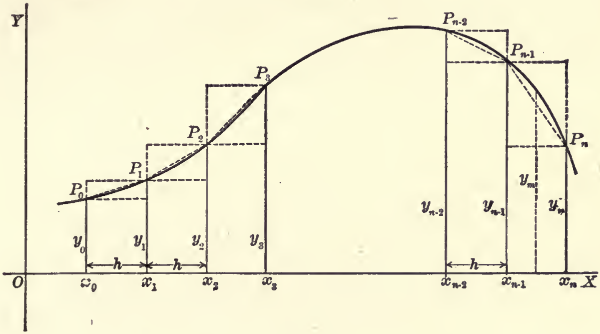


FIG. 101.

If, starting at P_n, we draw segments parallel to the x-axis through the points $P_n, P_{n-1}, \ldots, P_2, P_1$, the area enclosed by the rectangles thus formed is given by

$$A_R' = h\,(y_1 + y_2 + y_3 + \cdots + y_n).$$

It is evident that the smaller the interval h, the better the approximation to the required area.

(2) *Trapezoidal rule.* — If the chords $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$ are drawn, then the area enclosed by the trapezoids thus formed is

$$A_T = h\left(\frac{y_0 + y_1}{2}\right) + h\left(\frac{y_1 + y_2}{2}\right) + h\left(\frac{y_2 + y_3}{2}\right) + \cdots + h\left(\frac{y_{n-1} + y_n}{2}\right)$$
$$= h\left[\tfrac{1}{2}(y_0 + y_n) + y_1 + y_2 + \cdots + y_{n-1}\right].$$

This expression for the area is the average of the two expressions given by the rectangular rules. It is evident that the smaller the interval h and the flatter the curve, the better the approximation to the required area. If the curve is steep at either end or anywhere within the interval, the rule may be modified by subdividing the smaller interval into 2 or 4 parts; thus, subdividing the steep interval between x_{n-1} and x_n in Fig. 101

into 2 parts: $A_T = h\left(\dfrac{y_0 + y_1}{2}\right) + \cdots + \dfrac{h}{2}\left(\dfrac{y_{n-1} + y_m}{2}\right) + \dfrac{h}{2}\left(\dfrac{y_m + y_n}{2}\right),$

into 4 parts: $A_T = h\left(\frac{y_0+y_1}{2}\right) + \cdots + \frac{h}{4}\left(\frac{y_{n-1}+y_k}{2}\right) + \frac{h}{4}\left(\frac{y_k+y_m}{2}\right)$

$$+\frac{h}{4}\left(\frac{y_m+y_l}{2}\right)+\frac{h}{4}\left(\frac{y_l+y_n}{2}\right).$$

(3) *Simpson's rule.* — Let us pass arcs of parabolas through the points $P_0P_1P_2$, $P_2P_3P_4$, . . . , $P_{n-2}P_{n-1}P_n$. Let the equation of the parabola through $P_0P_1P_2$ be $y = ax^2 + bx + c$. Then the area bounded by the parabola, the x-axis, and the ordinates $x = x_0$ and $x = x_2$ is

$$A = \int_{x_0}^{x_2}(ax^2+bx+c)\,dx = \left|\frac{ax^3}{3}+\frac{bx^2}{2}+cx\right|_{x_0}^{x_2} = \frac{a}{3}(x_2^3 - x_0^3) + \frac{b}{2}(x_2^2 - x_0^2)$$

$$+ c(x_2 - x_0) = \frac{x_2 - x_0}{6}[2a(x_2^2 + x_2x_0 + x_0^2) + 3b(x_2+x_0) + 6c].$$

Now, $\quad y_0 = ax_0^2 + bx_0 + c, \quad y_2 = ax_2^2 + bx_2 + c, \quad h = \frac{x_2 - x_0}{2},$

$$y_1 = ax_1^2 + bx_1 + c = a\left(\frac{x_2+x_0}{2}\right)^2 + b\left(\frac{x_2+x_0}{2}\right) + c,$$

and we may easily verify that

$$A = \tfrac{1}{3}h(y_0 + 4y_1 + y_2).$$

If we have an even number of intervals and apply this formula to the successive areas under the parabolic arcs, we get

$$\begin{aligned}A_S &= \tfrac{1}{3}h(y_0+4y_1+y_2) + \tfrac{1}{3}h(y_2+4y_3+y_4) + \cdots + \tfrac{1}{3}h(y_{n-2}+4y_{n-1}+y_n)\\ &= \tfrac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)\\ &= \tfrac{1}{3}h[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})].\end{aligned}$$

To apply Simpson's rule we must divide the interval into an *even* number of parts, and the required area is approximately equal to the sum of the extreme ordinates, plus four times the sum of the ordinates with odd subscripts, plus twice the sum of the ordinates with even subscripts, all multiplied by one-third the common distance between the ordinates.

(4) *Durand's rule.** — If we have an even number of parts and apply Simpson's rule to the interval from x_1 to x_{n-1} and the Trapezoidal rule to the end intervals,

$$\begin{aligned}A = h\,[(\tfrac{1}{2}y_0 + \tfrac{1}{2}y_1) + (\tfrac{1}{3}y_1 + \tfrac{4}{3}y_2 + \tfrac{2}{3}y_3 + \cdots\\ + \tfrac{2}{3}y_{n-3} + \tfrac{4}{3}y_{n-2} + \tfrac{1}{3}y_{n-1}) + (\tfrac{1}{2}y_{n-1} + \tfrac{1}{2}y_n)].\end{aligned}$$

Applying Simpson's rule to the entire interval from x_0 to x_n,

$$A = h\,[\tfrac{1}{3}y_0 + \tfrac{4}{3}y_1 + \tfrac{2}{3}y_2 + \tfrac{4}{3}y_3 + \cdots + \tfrac{4}{3}y_{n-3} + \tfrac{2}{3}y_{n-2} + \tfrac{4}{3}y_{n-1} + \tfrac{1}{3}y_n].$$

Adding,

$$2A = h\,[\tfrac{5}{6}y_0 + \tfrac{13}{6}y_1 + 2y_2 + 2y_3 + \ldots + 2y_{n-3} + 2y_{n-2} + \tfrac{13}{6}y_{n-1} + \tfrac{5}{6}y_n].$$

* Given by Prof. Durand in Engineering News, Jan., 1894.

Hence,

$$A_D = h\left[\tfrac{5}{12}(y_0+y_n) + \tfrac{13}{12}(y_1+y_{n-1}) + y_2 + y_3 + \cdots + y_{n-2}\right]$$
$$= h\left[0.4(y_0+y_n) + 1.1(y_1+y_{n-1}) + y_2 + y_3 + \cdots + y_{n-2}\right].$$

Collecting our rules, we have

(1) $A_R = h(y_0 + y_1 + y_2 + \cdots + y_{n-1})$,

or $A_R' = h(y_1 + y_2 + y_3 + \cdots + y_n)$.

(2) $A_T = h[\tfrac{1}{2}(y_0+y_n) + y_1 + y_2 + \cdots + y_{n-1}]$.

(3) $A_S = \tfrac{1}{3}h[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})]$.

(4) $A_D = h[0.4(y_0+y_n) + 1.1(y_1+y_{n-1}) + y_2 + y_3 + \cdots + y_{n-2}]$.

102. Applications of approximate rules. — We shall give some examples illustrating the application of these rules.

1. *Area.* — Evaluate $\int_2^{10}\frac{dx}{x}$. This is equivalent to finding the area between the curve $y = 1/x$, the x-axis, and the ordinates $x = 2$ and $x = 10$. If we divide the interval into 8 parts, then $h = 1$; we have the table

x	2	3	4	5	6	7	8	9	10
y	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$

$$A_R = 1\left(\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots+\tfrac{1}{9}\right) = 1.8290;$$
$$A_R' = 1\left(\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\cdots+\tfrac{1}{10}\right) = 1.4290;$$
$$A_T = 1\left[\tfrac{1}{2}\left(\tfrac{1}{2}+\tfrac{1}{10}\right)+\tfrac{1}{3}+\cdots+\tfrac{1}{9}\right] = 1.6290;$$
$$A_S = \tfrac{1}{3}\left[\left(\tfrac{1}{2}+\tfrac{1}{10}\right)+4\left(\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{9}\right)+2\left(\tfrac{1}{4}+\tfrac{1}{6}+\tfrac{1}{8}\right)\right] = 1.6109;$$
$$A_D = 1\left[0.4\left(\tfrac{1}{2}+\tfrac{1}{10}\right)+1.1\left(\tfrac{1}{3}+\tfrac{1}{9}\right)+\tfrac{1}{4}+\tfrac{1}{5}+\cdots+\tfrac{1}{8}\right] = 1.6134.$$

By actual integration, $\int_2^{10}\frac{dx}{x} = \Big|\ln x\Big|_2^{10} = \ln 10 - \ln 2 = \ln 5 = 1.6094.$

We note that Simpson's rule gives the best approximation (within 0.1 % of the true value), with Durand's next.

If we take $h = \frac{1}{2}$,

$$A_T = \frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}+\frac{1}{10}\right)+\frac{1}{5/2}+\frac{1}{3}+\frac{1}{7/2}+\cdots+\frac{1}{19/2}\right] = 1.6144;$$
$$A_S = \frac{1}{6}\left[\left(\frac{1}{2}+\frac{1}{10}\right)+4\left(\frac{1}{5/2}+\frac{1}{7/2}+\cdots+\frac{1}{19/2}\right)+2\left(\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{9}\right)\right] = 1.6096.$$

Thus the Trapezoidal rule with 16 ordinates does not give the accuracy given by Simpson's rule with 8 ordinates.

2. *Area.* — The half-ordinates in feet of the mid-ship section of a vessel are

12.5, 12.8, 12.9, 13.0, 13.0, 12.8, 12.4, 11.8, 10.4, 6.8, 0.5,

and the ordinates are 2 feet apart; find the area of the whole section.

$$\tfrac{1}{2} A_T = 2\left[\tfrac{1}{2}(12.5 + 0.5) + 12.8 + \cdots + 6.8\right] = 224.8;$$

$$\tfrac{1}{2} A_S = \tfrac{2}{3}\left[(12.5 + 0.5) + 4(12.8 + 13.0 + 12.8 + 11.8 + 6.8) + 2(12.9 + 13.0 + 12.4 + 10.4)\right] = 226.1.$$

Hence, $\quad A_T = 449.6$ sq. ft., $\quad A_S = 452.2$ sq. ft.

3. *Work.* — Given the following data for steam

v	2	4	6	8	10
p	68.7	31.3	19.8	14.3	11.3

where v is the volume in cu. ft. per pound and p is the pressure in pounds per sq. in.; find the work done by the piston.

Work $= \int_2^{10} p\, dv$; this is equivalent to finding the area under the curve obtained by plotting (v, p).

$$W_T = 2\left[\tfrac{1}{2}(68.7 + 11.3) + 31.3 + 19.8 + 14.3\right] = 210.80;$$

$$W_S = \tfrac{2}{3}\left[(68.7 + 11.3) + 4(31.3 + 14.3) + 2(19.8)\right] = 201.33.$$

By the methods of Chapter VI we find the empirical formula connecting v and p to be $pv^{1.12} = 148$, and hence,

$$W = \int_2^{10} p\, dv = 148 \int_2^{10} v^{-1.12}\, dv = 148 \left| \frac{v^{-0.12}}{-0.12} \right|_2^{10} = 199.31.$$

This last value differs from the value given by Simpson's rule by about 1%.

4. *Mean effective pressure. Indicator diagram.* Fig. 102*a* is a reproduction of an indicator diagram; to find the mean effective pressure.

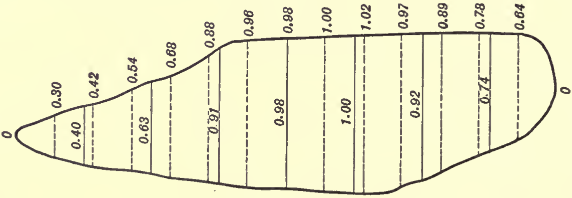


FIG. 102*a*.

The mean effective pressure P is the area of the diagram divided by the length of the diagram, since the area represents the effective area of the piston in sq. in. and the length represents the length of the stroke in ft. Since the total area enclosed by the curve is the difference between the area bounded by a horizontal axis, the end ordinates, and the upper part of the curve, and the area bounded by the same straight lines and the lower part of the curve, we need merely measure the lengths of the ordinates within the curve. The diagram is 3.5 ins. long. We divide the interval into 8 parts; then $h = \frac{7}{16}$, and we measure the ordinates

0, 0.40, 0.63, 0.91, 0.98, 1.00, 0.92, 0.74, 0.

$A_T = \frac{7}{16}[0.40 + 0.63 + \cdots + 0.74] = 2.44;$
$A_S = \frac{7}{48}[4(0.40 + 0.91 + 1.00 + 0.74) + 2(0.63 + 0.98 + 0.92)] = 2.52.$

Hence, $$P = \frac{A_S}{3.5} = \frac{2.52}{3.5} = 0.72.$$

We divide the interval into 14 parts; then $h = \frac{1}{4}$, and we measure the ordinates

0, 0.30, 0.42, 0.54, 0.68, 0.88, 0.96, 0.98, 1.00, 1.02, 0.97, 0.89, 0.78, 0.64, 0.

$A_T = \frac{1}{4}[0.30 + 0.42 + \cdots + 0.64] = 2.52.$
$A_S = \frac{1}{12}[4(0.30+0.54+ \cdots +0.64)+2(0.42+0.68+ \cdots +0.78)] = 2.55.$

Hence, $$P = \frac{A_S}{3.5} = \frac{2.55}{3.5} = 0.73.$$

We note that A_S with 9 ordinates has the same value as A_T with 15 ordinates.

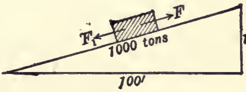


FIG. 102b.

5. *Velocity.* — Given a weight of 1000 tons sliding down a 1% grade (Fig. 102*b*) with a frictional resistance of 10 lbs. per ton at all speeds. The total resistance is 30,000 lbs. (a frictional resistance of 10,000 lbs. and a grade resistance of 20,000 lbs.). Let the following table express the accelerated force F as a function of the time t in seconds:

t	0	100	200	300	400	500	600	700	800	900	1000
F	20,000	19,000	16,000	11,000	5000	−1000	−5000	−8500	−11,000	−13,000	−15,000

Find the velocity acquired by the body in 1000 seconds.

Since $F = m \times a$, and $m = \frac{2,000,000}{g} = \frac{1,000,000}{16.1}$,

therefore, $a = \frac{F}{m} = \frac{16.1\,F}{1,000,000}$; and $\frac{dv}{dt} = a$, hence, $v = \int_0^{1000} a\,dt.$

We form a table for the acceleration a.

t	0	100	200	300	400	500	600	700	800	900	1000
a	0.322	0.306	0.258	0.177	0.081	−0.016	−0.081	−0.137	−0.177	−0.209	−0.242

Here, $h = 100$, so that

$v_T = 100[\frac{1}{2}(0.322-0.242)+(0.306+0.258+ \cdots -0.209)] = 24.2$ ft. per sec.
$v_S = \frac{100}{3}[(0.322 - 0.242) + 4(0.306 + 0.177 - 0.016 - 0.137 - 0.209)$
$+ 2(0.258 + 0.081 - 0.081 - 0.177)] = 24.2$ ft. per sec.

6. *Volume.* — If S_x is the area of a cross-section of a solid made by a plane perpendicular to the x-axis, then the volume of the solid included between the planes x_0 and x_n is $V = \int_{x_0}^{x_n} S_x\,dx$. In order to integrate, we must know the analytical expression for S_x as a function of x. Otherwise we employ the approximate formulas; the values of S_x are the ordinates and h is the common distance between the cutting planes.

A buoy is in the form of a solid of revolution with its axis vertical, and D is the diameter in ft. at a depth p ft. below the surface of the water.

p	0	0.3	0.6	0.9	1.2	1.5	1.8
D	6.00	5.90	5.80	5.55	5.25	4.70	4.20
D^2	36.00	34.81	33.64	30.80	27.56	22.09	17.64

Find the weight of water displaced by the buoy (1 cu. ft. of sea water weighs 64.11 lbs.).

Here, $$V = \int_0^{1.8} \frac{\pi}{4} D^2\, dp, \quad \text{and} \quad h = 0.3,$$

hence, $$V_S = \frac{0.3}{3}\frac{\pi}{4}[(36.00 + 17.64) + 4\,(34.81 + 30.80 + 22.09) + 2\,(33.64 + 27.56)] = 41.38 \text{ cu. ft.},$$

and the weight of water displaced = 2652.87 lbs.

The areas in sq. ft. of the sections of a ship below the load-water plane and 3 ft. apart are

7500, 7150, 6640, 5680, 4225, 2430, 260,

where the load-water plane has an area of 7500 sq. ft. Find the displacement in tons (35 cu. ft. of sea water weigh 1 ton).

$$V_T = 3\,[\tfrac{1}{2}(7500+260)+(7150+6640+5680+4225+2430)] = 90{,}015 \text{ cu. ft.}$$
$$V_S = \tfrac{3}{3}[(7500+260)+4(7150+5680+2430)+2(6640+4225)] = 90{,}530 \text{ cu. ft.}$$

Hence, the displacement is 2572 tons by the Trapezoidal rule and 2587 tons by Simpson's rule.

7. *Moment of inertia.* — The moments of inertia of an area about the axes are

$$J_x = \int_{x_0}^{x_n} \tfrac{1}{3} y^3\, dx, \qquad J_y = \int_{x_0}^{x_n} x^2 y\, dx.$$

The evaluation of these integrals is equivalent to finding the areas under the curves with $\frac{1}{3}y^3$ or x^2y as ordinates and x as abscissas.

The half-ordinates in ft. of the mid-ship section of a vessel are

12.5, 12.8, 12.9, 13.0, 13.0, 12.8, 12.4, 11.8, 10.4, 6.8, 0.5,

and the ordinates are 2 ft. apart. Find the moment of inertia of the entire section about the axis.

Here, $J_x = 2\int_0^{20} \frac{1}{3} y^3\, dx$, $h = 2$, and the values of y^3 are

1953.1, 2097 2, 2146.7, 2197.0, 2197.0, 2097.2, 1906.6, 1643.0, 1124.9, 314.4, 0.1,

and applying Simpson's rule,

$$J_x = \tfrac{2}{3}(\tfrac{2}{3})[(1953.1 + 0.1) + 4\,(2097.2 + \cdots + 314.4) + 2\,(2146.7 + \cdots + 1124.9)] = 22{,}266.1.$$

8. *Pressure and center of pressure.* — The pressure on a plane area perpendicular to the surface of the liquid, between depths x_0 and x_n, is $p = w\int_{x_0}^{x_n} xy\,dx$, where w is the weight of the liquid per unit volume, y is the width of the area at a depth x beneath the surface. The depth of the center of pressure of such an area is given by $\bar{x} = \dfrac{\int_{x_0}^{x_n} x^2y\,dx}{\int_{x_0}^{x_n} xy\,dx}$. All these integrals can be evaluated approximately.

9. *Center of gravity.* — The coördinates of the center of gravity of an area are

$$\bar{x} = \frac{\int xy\,dx}{\int y\,dx} = \frac{\text{Moment about } OY}{\text{Area}},\ \bar{y} = \frac{\int \frac{1}{2}y^2\,dx}{\int y\,dx} = \frac{\text{Moment about } OX}{\text{Area}}.$$

The half-ordinates in ft. of the mid-ship section of a vessel are

12.5, 12.8, 12.9, 13.0, 13.0, 12.8, 12.4, 11.8, 10.4, 6.8, 0.5,

and the ordinates are 2 ft. apart. Find the center of gravity of the section.

$$\bar{x} = \frac{\int_0^{20} xy\,dx}{\int_0^{20} y\,dx} = \frac{\text{Moment about } OY}{\text{Area}},$$

and applying Simpson's rule to the table,

x	0	2	4	6	8	10	12	14	16	18	20
y	12.5	12.8	12.9	13.0	13.0	12.8	12.4	11.8	10.4	6.8	0.5
xy	0	25.6	51.6	78.0	104.0	128.0	148.8	165.2	166.4	122.4	10.0

$$M_S = \tfrac{2}{3}[(0 + 10.0) + 4(25.6 + \cdots + 122.4) + 2(51.6 + \cdots + 166.4)]$$
$$= 2018.9.$$
$$A_S = \tfrac{2}{3}[(12.5 + 0.5) + 4(12.8 + \cdots + 6.8) + 2(12.9 + \cdots + 10.4)]$$
$$= 226.1.$$

Hence, $$\bar{x} = \frac{2018.9}{226.1} = 8.93 \text{ ft.}$$

103. General formula for approximate integration. — We may derive a general formula for approximate integration by integrating any of the formulas of interpolation. Thus, Newton's formula (p. 215),

$$y_n = y_0 + na_0 + \frac{n(n-1)}{\underline{|2}}b_0 + \cdots + \frac{n(n-1)\ldots(n-k+1)}{\underline{|k}}k_0,$$

where $x = x_0 + nh$, is true for all values of n if some order of differences is constant or approximately constant. Multiplying by dn and integrating term by term between the limits 0 and n, we have

$$\int_0^n y_n\,dn = y_0\int_0^n dn + a_0\int_0^n n\,dn + \frac{b_0}{\underline{|2}}\int_0^n n\,(n-1)\,dn$$
$$+\frac{c_0}{\underline{|3}}\int_0^n n\,(n-1)\,(n-2)\,dn + \cdots.$$

Since $x = x_0 + nh$, therefore, $n = \dfrac{x - x_0}{h}$ and $dn = \dfrac{1}{h}dx$. Hence,

$$\int_{x_0}^{x_n} y\,dx = h\left[ny_0 + \frac{n^2}{2}a_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{b_0}{\underline{|2}} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{c_0}{\underline{|3}}\right.$$
$$\left.+\left(\frac{n^5}{5} - \frac{3\,n^4}{2} + \frac{11\,n^3}{3} - 3\,n^2\right)\frac{d_0}{\underline{|4}} + \left(\frac{n^6}{6} - 2\,n^5 + \frac{35\,n^4}{4} - \frac{50\,n^3}{3} + 12\,n^2\right)\frac{e_0}{\underline{|5}} + \cdots\right].$$

Thus, if the differences after some order, as the kth, are negligible, we may use this formula to get the approximate area between the curve, the x-axis, and the ordinates $x = x_0$ and $x = x_n$. The process is equivalent to approximating the equation of the curve by a polynomial of the kth degree. The differences $a_0, b_0, c_0, \ldots$ are those which occur in a line through y_0 parallel to the upper side of the triangle in the scheme on p. 210. Similar integration formulas can be derived from the other interpolation formulas.

If the interval from x_0 to x_n is large, it is well to divide this into smaller intervals, apply the formula to each of the smaller intervals, and add the results. In this way we may derive the formulas of Art. 101 and similar formulas as special cases of the above general formula.

Let us first note that by means of the rule for the formation of the successive differences of a function (p. 210) we may express the differences $a_0, b_0, c_0, \ldots$ in terms of $y_0, y_1, y_2, \ldots$. Thus,

$$\begin{aligned}
a_0 &= y_1 - y_0,\\
b_0 &= a_1 - a_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2\,y_1 + y_0,\\
c_0 &= b_1 - b_0 = [(y_3 - 2\,y_2 + y_1) - (y_2 - 2\,y_1 + y_0)] = y_3 - 3\,y_2 + 3\,y_1 - y_0,\\
d_0 &= y_4 - 4\,y_3 + 6\,y_2 - 4\,y_1 + y_0,\\
e_0 &= y_5 - 5\,y_4 + 10\,y_3 - 10\,y_2 + 5\,y_1 - y_0,\\
&\quad \cdots\cdots\cdots\cdots\\
k_0 &= y_k - ky_{k-1} + \frac{k\,(k-1)}{\underline{|2}}y_{k-2} - \cdots
\end{aligned}$$

where the coefficients in the right members of these equations are the binomial coefficients, taken alternately plus and minus.

(1) Let $n = 1$ and $b_0, c_0, \ldots$ all zero, *i.e.*, approximate the curve (Fig. 101*a*) from x_0 to x_1 by a straight line, $y = A + Bx$. Then

$$\int_{x_0}^{x_1} y\,dx = h\,[y_0 + \tfrac{1}{2}\,a_0] = h\,[y_0 + \tfrac{1}{2}\,(y_1 - y_0)] = h\left[\frac{y_0 + y_1}{2}\right].$$

Applying this result to each interval and adding, we get the *Trapezoidal rule:*

$$A_T = \int_{x_0}^{x_n} y\,dx = h\,[\tfrac{1}{2}\,(y_0 + y_n) + y_1 + y_2 + \cdots + y_{n-1}].$$

(2) Let $n = 2$ and $c_0, d_0, \ldots$ all zero, *i.e.*, approximate the curve (Fig. 101*a*) from x_0 to x_2 by a parabola, $y = A + Bx + Cx^2$. Then

$$\int_{x_0}^{x_2} y\,dx = h\,[2\,y_0 + 2\,a_0 + \tfrac{1}{3}\,b_0] = h\,[2\,y_0 + 2\,(y_1 - y_0) + \tfrac{1}{3}\,(y_2 - 2\,y_1 + y_0)]$$
$$= \frac{h}{3}\,[y_0 + 4\,y_1 + y_2].$$

Applying this result to an even number of intervals, two at a time, and adding, we get *Simpson's rule:*

$$A_S = \int_{x_0}^{x_n} y\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})].$$

(3) Let $n = 3$ and $d_0, e_0, \ldots$ all zero, *i.e.*, approximate the curve (Fig. 101*a*) from x_0 to x_3 by a parabola of the 3d degree, $y = A + Bx + Cx^2 + Dx^3$. Then

$$\int_{x_0}^{x_3} y\,dx = h\,[3\,y_0 + \tfrac{9}{2}\,a_0 + \tfrac{9}{4}\,b_0 + \tfrac{3}{8}\,c_0] = h\,[3\,y_0 + \tfrac{9}{2}\,(y_1 - y_0) + \tfrac{9}{4}\,(y_2 - 2\,y_1 + y_0)$$
$$+ \tfrac{3}{8}\,(y_3 - 3\,y_2 + 3\,y_1 - y_0)] = \tfrac{3}{8}\,h\,[y_0 + 3\,y_1 + 3\,y_2 + y_3].$$

Applying this result to n intervals, where n is a multiple of 3, and adding, we get *Simpson's* three-eighths rule:

$$A_S' = \int_{x_0}^{x_n} y\,dx = \tfrac{3}{8}\,h\,[y_0 + 2\,(y_3 + y_6 + y_9 + \cdots) + 3\,(y_1 + y_2 + y_4 + y_5 + \cdots)].$$

(4) Let $n = 6$ and the differences beyond the 6th order negligible, *i.e.*, approximate the curve (Fig. 101*a*) from x_0 to x_6 by a parabola of the 6th degree, $y = A + Bx + Cx^2 + \cdots + Hx^6$. Then

$$\int_{x_0}^{x_6} y\,dx = h\,[6\,y_0 + 18\,a_0 + 27\,b_0 + 24\,c_0 + \tfrac{123}{10}\,d_0 + \tfrac{33}{10}\,e_0 + \tfrac{41}{140}\,f_0].$$

Substituting the values of $a_0, b_0, \ldots, f_0$ in terms of the y's and replacing $\tfrac{41}{140}\,f_0$ by $\tfrac{3}{10}\,f_0$, thus neglecting $\tfrac{1}{140}\,f_0$ which will be fairly small, we get *Weddle's rule:*

$$A_W = \int_{x_0}^{x_6} y\,dx = \tfrac{3}{10}\,h\,[y_0 + 5\,y_1 + y_2 + 6\,y_3 + y_4 + 5\,y_5 + y_6].$$

We may apply this rule to n intervals where n is a multiple of 6.

Example. Apply the approximate rules (1) to (4) to evaluate $\int_2^{2.6}\frac{dx}{x}$.

We divide the interval into 6 equal parts, so that $h = 0.1$. From the table

x	2	2.1	2.2	2.3	2.4	2.5	2.6
y	$\frac{1}{2}$	$\frac{1}{2.1}$	$\frac{1}{2.2}$	$\frac{1}{2.3}$	$\frac{1}{2.4}$	$\frac{1}{2.5}$	$\frac{1}{2.6}$

$$A_T = 0.1\left[\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2.6}\right)+\frac{1}{2.1}+\frac{1}{2.2}+\frac{1}{2.3}+\frac{1}{2.4}+\frac{1}{2.5}\right] = 0.2624493;$$

$$A_S = \frac{0.1}{3}\left[\left(\frac{1}{2}+\frac{1}{2.6}\right)+4\left(\frac{1}{2.1}+\frac{1}{2.3}+\frac{1}{2.5}\right)+2\left(\frac{1}{2.2}+\frac{1}{2.4}\right)\right] = 0.2623644;$$

$$A_S' = \frac{3}{8}(0.1)\left[\frac{1}{2}+3\left(\frac{1}{2.1}\right)+3\left(\frac{1}{2.2}\right)+2\left(\frac{1}{2.3}\right)+3\left(\frac{1}{2.4}\right)\right.$$
$$\left.+3\left(\frac{1}{2.5}\right)+\frac{1}{2.6}\right] = 0.2623645;$$

$$A_W = \frac{3}{10}(0.1)\left[\frac{1}{2}+5\left(\frac{1}{2.1}\right)+\frac{1}{2.2}+6\left(\frac{1}{2.3}\right)+\frac{1}{2.4}+5\left(\frac{1}{2.5}\right)+\frac{1}{2.6}\right] = 0.2623643.$$

By integration, $A = \int_2^{2.6}\frac{dx}{x} = \Big|\ln x\Big|_2^{2.6} = \ln 2.6 - \ln 2 = \ln 1.3 = 0.2623637.$

A_T agrees with A to 4 decimals, while A_S, A_S', and A_W agree about equally well with A to 6 decimals.

104. Numerical differentiation. — We are to find the slope of the curve $y = f(x)$ at any point when the curve is drawn or a table of values of equidistant ordinates are given, *i.e.*, we are to find $\frac{dy}{dx}$ when the analytical form of the function is unknown. Graphically, we must construct the tangent line to the curve at the given point. The exact or even approximate construction of the tangent line to a curve (except for the parabola) is difficult and inaccurate.*

We may derive an expression for $\frac{dy}{dx}$ by differentiating Newton's interpolation formula. Newton's formula

$$y_n = y_0 + na_0 + \frac{n(n-1)}{\underline{|2}}b_0 + \cdots + \frac{n(n-1)\ldots(n-k+1)}{\underline{|k}}k_0,$$

is true for all values of n if some order of differences, as the kth, is constant or approximately constant.

Since $x = x_0 + nh$, therefore, $dx = h\,dn$, and $\frac{dy}{dx} = \frac{1}{h}\frac{dy}{dn}$, $\frac{d^2y}{dx^2} = \frac{1}{h^2}\frac{d^2y}{dn^2}$.

* See Art. 106 on graphical differentiation.

Hence,

$$\frac{dy}{dx}=\frac{1}{h}\left[a_0+(2n-1)\frac{b_0}{\lfloor 2}+(3n^2-6n+2)\frac{c_0}{\lfloor 3}+(4n^3-18n^2+22n-6)\frac{d_0}{\lfloor 4}+\cdots\right],$$

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left[b_0+(n-1)c_0+(6n^2-18n+11)\frac{d_0}{12}+\cdots\right].$$

The values of these coefficients are tabulated for values of n between 0 and 1 at intervals of 0.01.*

For the tabulated values $x_0, x_1, \ldots, x_n$, we have $n = 0$, so that for these values of x we have the simpler formulas

$$\frac{dy}{dx}=\frac{1}{h}\left[a_0-\frac{1}{2}b_0+\frac{1}{3}c_0-\frac{1}{4}d_0+\cdots\right],$$

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left[b_0-c_0+\frac{11}{12}d_0+\cdots\right].$$

If the value of x for which $\frac{dy}{dx}$ is required is near the end of the table, we may use similar formulas derived from the modified Newton's formula for end-interpolation (p. 217).

Example. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = 3$ and $x = 3.3$ from table (1) on p. 211 and check the results by differentiating $y = x^3$.

Since $x = 3$ is a tabulated value we apply the second set of formulas:

$$\frac{dy}{dx}=\left[37-\frac{1}{2}(24)+\frac{1}{3}(6)\right]=27;\quad \frac{d^2y}{dx^2}=[24-6]=18.$$

From $\quad y = x^3,\quad \frac{dy}{dx}=3x^2=27,\quad \frac{d^2y}{dx^2}=6x=18.$

For $x = 3.3$ we apply the first set of formulas, where $a_0 = 37$, $b_0 = 24$, $c_0 = 6$, $n = 0.3$. Then

$$\frac{dy}{dx}=\left[37+(-0.4)\frac{24}{2}+(0.47)\frac{6}{6}\right]=32.67;\ \frac{d^2y}{dx^2}=\left[24+(-0.7)\,6\right]=19.8.$$

From $\quad y = x^3,\quad \frac{dy}{dx}=3x^2=32.67,\quad \frac{d^2y}{dx^2}=6x=19.8.$

Example. Rate of change.—The following table gives the results of observation; θ is the observed temperature in degrees Centigrade of a vessel of cooling water, t is the time in minutes from the beginning of observation.

t	0	1	2	3	4	5
θ	92.0	85.3	79.5	74.5	70.2	67.0

To find the approximate rate of cooling when $t = 1$ and $t = 2.5$.

* See Rice, *Theory and Practice of Interpolation.*

From the table of differences

t	θ	Δ^1	Δ^2	Δ^3
0	92.0			
		−6.7		
1	85.3		0.9	
		−5.8		−0.1
2	79.5		0.8	
		−5.0		−0.1
3	74.5		0.7	
		−4.3		0.4
4	70.2		1.1	
		−3.2		
5	67.0			

when $t = 1$, $n = 0$ and $\dfrac{d\theta}{dt} = \left[-5.8 - \frac{1}{2}(0.8) + \frac{1}{3}(-0.1)\right] = -6.23;$

when $t = 2.5$, $n = 0.5$ and $\dfrac{d\theta}{dt} = \left[-5.0 + 0 + (-0.25)\left(\frac{+0.4}{6}\right)\right] = -5.02.$

Example. Maximum and minimum.—The following table gives the results of measurements made on a magnetization curve of iron; B is the number of kilolines per sq. cm., μ is the permeability (Fig. 104).

B	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
μ	370	570	730	865	985	1090	1175	1245	1295	1330	1340	1320	1250	1120	930	725

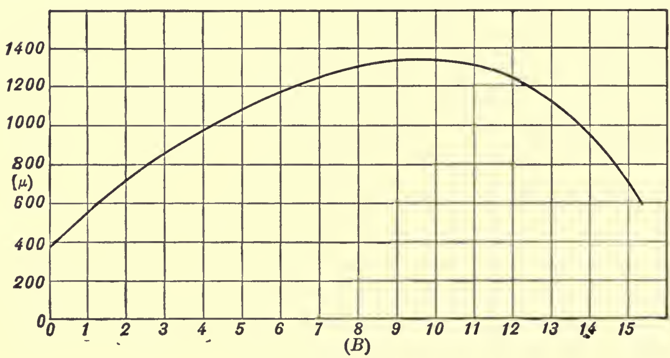


FIG. 104.

To find the maximum permeability. In Fig. 104 the maximum permeability appears to be in the neighborhood of $B = 10$. We therefore tabulate the differences of μ in the neighborhood of $B = 10$.

B	μ	Δ^1	Δ^2	Δ^3
9	1330			
		10		
10	1340		−30	
		−20		−20
11	1320		−50	
		−70		−10
12	1250		−60	
		−130		
13	1120			

For values of B between $B = 9$ and $B = 10$, we have

$$\frac{d\mu}{dB} = \left[10 + (2n-1)\left(-\frac{30}{2}\right) + (3n^2-6n+2)\left(-\frac{20}{6}\right)\right] = \frac{5}{3}(11-6n-6n^2).$$

For a maximum, $\dfrac{d\mu}{dB} = 0$, hence $6n^2 + 6n - 11 = 0$, and $n = 0.94$. Therefore, $B = B_0 + nh = 9.94$.

We find the corresponding value of μ by the interpolation formula,

$$\mu = 1330 + (0.94)(10) + (0.0282)(-30) + (0.0100)(-20) = 1340.$$

If we take account of Δ^1 and Δ^2 only, we get

$$\frac{d\mu}{dB} = 10 + (2n-1)\left(-\frac{30}{2}\right) = 0, \quad \text{or} \quad n = \frac{5}{6} = 0.83, \quad \text{and} \quad B = 9.83.$$

Then $\quad \mu = 1330 + (0.83)(10) + (0.0275)(-30) = 1337.5.$

105. Graphical integration. — Let us find the value of the definite integral $\int_a^b f(x)\,dx$ or the area under the curve $y = f(x)$ by graphical methods. We draw the curve $y = f(x)$ (Fig. 105*a*) and along the ordinate at $P\ (x, y)$ erect the ordinate y' whose value is a measure of the area under

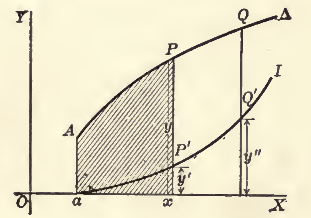


FIG. 105*a*.

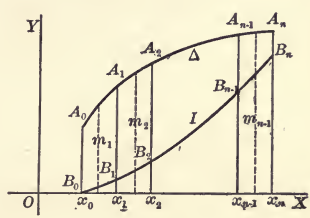


FIG. 105*b*.

the curve $y = f(x)$ from the initial point $A\ (x = a)$ to the point P, *i.e.*, $y' = \int_a^x f(x)\,dx$. Thus for every point $P\ (x, y)$ we have a corresponding point $P'\ (x, y')$. The curve traced by the point P' (marked I in the figure) is called the *integral curve* and the curve traced by the point P (marked Δ in the figure) is called the *derivative curve*. Evidently, if P and Q are two points on the Δ-curve and P' and Q' are their corresponding points on the I-curve, the difference of the ordinates of P' and Q', $y'' - y'$, is a measure of the area under the arc PQ.

The practical construction of the integral curve consists of the following steps (Fig. 105*b*).

(1) Divide the interval from x_0 to x_n into n equal or unequal intervals and erect the ordinates $y_0, y_1, \ldots, y_n$.

(2) Measure the areas $x_0A_0A_1x_1 = y_1'$, $x_0A_0A_2x_2 = y_2'$, . . . , $x_0A_0A_nx_n = y_n'$. These areas may be found by means of a planimeter or by the construction of the mean ordinates. Thus, the area $x_0A_0A_1x_1$ is equal to the area of a rectangle whose base is x_0x_1 and whose altitude is the mean ordinate m_1 within that area. Similarly, the area $x_1A_1A_2x_1$ is equal to the area of a rectangle whose base is x_1x_2 and whose altitude is the mean ordinate m_2 within that area. Estimate the mean ordinates $m_1, m_2, m_3, \ldots, m_n$ within the successive sections. Then

$$y_1' = m_1\,(x_0x_1),\quad y_2' = y_1' + m_2\,(x_1x_2),\quad y_3' = y_2' + m_3\,(x_2x_3),\ \ldots,$$
$$y_n' = y_{n-1}' + m_n\,(x_{n-1}x_n).$$

If the intervals are all equal, *i.e.*, $x_0x_1 = x_1x_2 = \ldots = x_{n-1}x_n = \Delta x$, then $y' = \Sigma m\Delta x$. (We shall later give a more exact construction for the mean ordinate.)

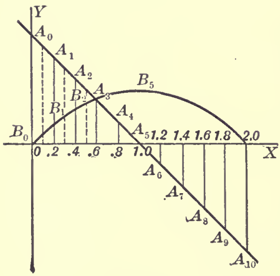


FIG. 105*c*.

(3) At $x_1, x_2, x_3, \ldots, x_n$ erect ordinates $x_1B_1, x_2B_2, \ldots, x_nB_n$ equal respectively to $y_1', y_2', \ldots, y_n'$, and draw a smooth curve through the points $B_0, B_1, B_2, \ldots, B_n$. This last curve will approximate the required integral curve.

Example. Construct the integral curve of the straight line $y = 1 - x$ between $x = 0$ and $x = 2$. (Fig. 105*c*.)

Divide the interval from $x = 0$ to $x = 2$ into 10 equal parts and erect the ordinates given in the table; here, $\Delta x = 0.2$.

x	y	m	$m\Delta x$	$y' = \Sigma m\Delta x$
0	1			0
0.2	0.8	0.9	0.18	0.18
0.4	0.6	0.7	0.14	0.32
0.6	0.4	0.5	0.10	0.42
0.8	0.2	0.3	0.06	0.48
1.0	0	0.1	0.02	0.50
1.2	−0.2	−0.1	−0.02	0.48
1.4	−0.4	−0.3	−0.06	0.42
1.6	−0.6	−0.5	−0.10	0.32
1.8	−0.8	−0.7	−0.14	0.18
2.0	−1.0	−0.9	−0.18	0

It is evident that the mean ordinate in each section is merely one-half the sum of the end ordinates, so that the values of m are easily found. Erect the ordinates y' and draw a smooth curve through the ends of the ordinates. The curve will approximate the parabola $y' = \int_0^x (1 - x)\,dx = x - \frac{1}{2}x^2$.

Example. The following table gives the accelerations a of a body sliding down an inclined plane at various times t, in seconds. To find the velocity and distance traversed at any time, if the initial velocity and initial distance are zero.

t	0	100	200	300	400	500	600	700	800	900	1000
a	0.320	0.304	0.256	0.176	0.080	−0.016	−0.080	−0.136	−0.176	−0.208	−0.240

Since $v = \int a\, dt$ and $s = \int v\, dt$, the time-velocity curve is the integral curve of the time-acceleration curve, and the time-distance curve is in turn the integral curve of the time-velocity curve.

In Fig. 105d, we have plotted t as abscissas and a as ordinates. The units chosen are 1 in. = 100 sec., and 1 in. = 0.16 ft. per sec. per sec.

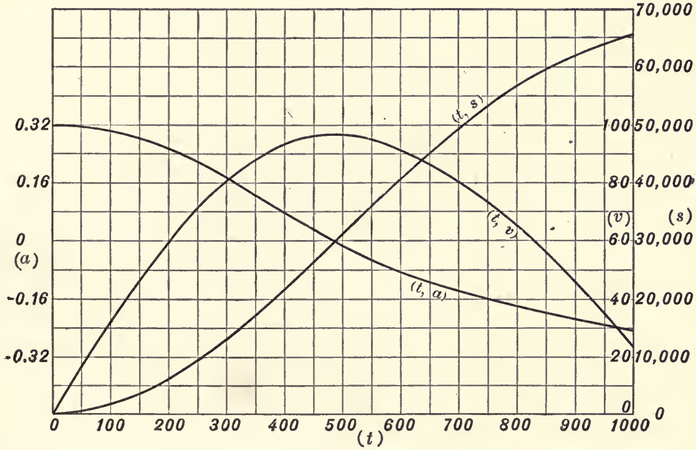


FIG. 105d.

t	a	avg. acc. a_m	$a_m \Delta t$	$v = \Sigma a_m \Delta t$	avg. vel. v_m	$v_m \Delta t$	$s = \Sigma v_m \Delta t$
0	0.320			0			0
		0.312	31.2		15.6	1560	
100	0.304			31.2			1,560
		0.280	28.0		45.2	4520	
200	0.256			59.2			6,080
		0.216	21.6		70.0	7000	
300	0.176			80.8			13,080
		0.128	12.8		87.2	8720	
400	0.080			93.6			21,800
		0.032	3.2		95.2	9520	
500	−0.016			96.8			31,320
		−0.048	− 4.8		94.4	9440	
600	−0.080			92.0			40,760
		−0.108	−10.8		86.6	8660	
700	−0.136			81.2			49,420
		−0.156	−15.6		73.4	7340	
800	−0.176			65.6			56,760
		−0.192	−19.2		56.0	5600	
900	−0.208			46.4			62,360
		−0.224	−22.4		35.2	3520	
1000	−0.240			24.0			65,880

In each interval of 100 sec. we have estimated the mean acceleration as the average of the accelerations at the beginning and end of the interval; thus, in the first interval, $a_m = \frac{1}{2}(0.320 + 0.304) = 0.312$. This is equivalent to replacing the arcs of the curve by their chords or to finding the area by the trapezoidal rule. Since the initial velocity is zero, the (t, v) curve joins $t = 0$, $v = 0$ with $t = 100$, $v = 31.2$, etc. We have drawn the (t, v) curve with a unit of 1 in. = 20 ft. sec.

In each interval of 100 sec. we have estimated the mean velocity as the average of the velocities at the beginning and end of the interval; thus in the first interval, $v_m = \frac{1}{2}(0 + 31.2) = 15.6$. Since the initial distance is zero, the (t, s) curve is drawn through the points $t = 0$, $s = 0$, $t = 100$, $s = 1560$, etc. The unit chosen is 1 in. = 10,000 ft.

The tables for v and s give the velocity and distance at the end of each 100 seconds, and we may interpolate graphically or numerically for the velocity and distance at any time between $t = 0$ and $t = 1000$.

In the foregoing discussion the accuracy of the construction of the integral curve depends largely upon the construction of the mean ordinates in the successive intervals. If the intervals are very small, we may get the required degree of accuracy by replacing the arcs by their chords and taking for the mean ordinate the average of the end ordinates.

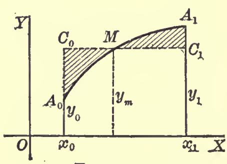


FIG. 105*e*.

The approximation of the mean ordinate for the arc A_0A_1 (Fig. 105*e*) is equivalent to finding a point M on the arc such that the area under the horizontal C_0C_1 through M is equal to the area under the arc A_0A_1 or such that the shaded areas A_0C_0M and A_1C_1M are equal. By means of a strip of celluloid and with a little practice the eye will find the position of M quite accurately, for the eye is very sensitive to differences in small areas.

We may draw the integral curve by a purely graphical process. Let us first consider the case when the derivative curve is the straight line AB parallel to the x-axis (Fig. 105*f*). Choose a fixed point S at any convenient distance a to the left of O. Extend AB to the point K on the y-axis and draw SK. Through A' (the projection of A on the x-axis) draw a line parallel to SK cutting the vertical through B in B'. Then, the oblique line $A'B'$ is the integral curve of the horizontal line AB. For, if P and P' are two corresponding points, then

$$y' : A'Q = y_0 : a, \text{ or } y' = \frac{1}{a}(y_0 \times A'Q) = \frac{1}{a} \times (\text{area under } AP).$$

Similarly, for another horizontal CD, with C and B in the same vertical line, extend CD to the point L on the y-axis and draw SL; through B'' draw a line parallel to SL cutting the vertical through D in C''; then, the oblique line $B''C''$ is the integral curve of the horizontal CD. Finally,

draw $B'C'$ parallel to $B''C''$ or to SL; then the broken oblique line $A'B'C'$ is the integral curve of the broken horizontal line $ABCD$.

Consider, now, any curve. Divide the interval from x_0 to x_n into n parts and erect the ordinates (Fig. 105g). Through A_0, A_1, A_2, . . . , draw short horizontal lines. Cut the arc A_0A_1 by a vertical line making

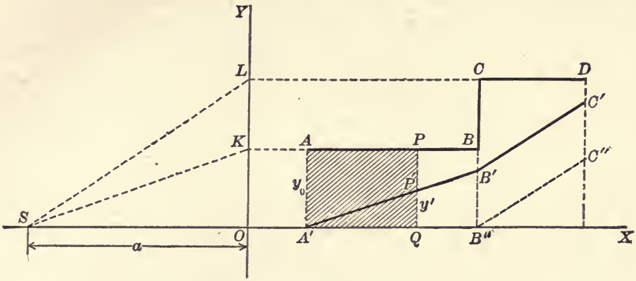


FIG. 105f.

the small areas bounded by this vertical, the arc, and the horizontals through A_0 and A_1, equal. Proceed similarly for the succeeding arcs. Then construct the integral curve of the stepped line by the method explained above. Choose a point S at a convenient distance a to the left of O and join S with the points C_0, C_1, C_2, . . . , in which the extended

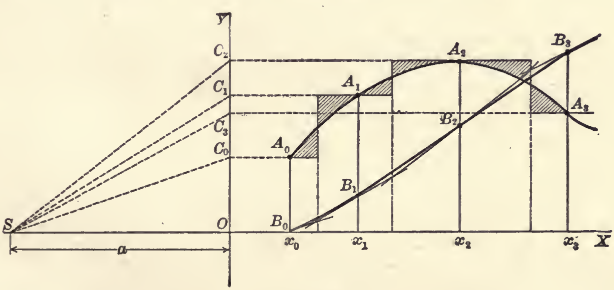


FIG. 105g.

horizontals cut the y-axis. Then, starting at B_0, draw a line through B_0 parallel to SC_0 until it cuts the first vertical; through this point draw a line parallel to SC_1 until it cuts the second vertical, etc. The points where the resulting broken line cuts the ordinates at A_0, A_1, A_2, . . . , *i.e.*, the points B_0, B_1, B_2, . . . , are points on the required integral curve; for at each of the points A_0, A_1, A_2, . . . , the area under the curve from

A_0 to that point is equal to the area under the stepped line; so that a smooth curve through the points B_0, B_1, B_2, . . . will be the required integral curve.

Since $y' = \frac{1}{a}\int y\,dx$, therefore, $\frac{dy'}{dx} = \frac{1}{a}y$, so that the slope of the integral curve at any point is proportional to the ordinate of the derivative curve at the corresponding point. Furthermore, by the construction, the slopes of the oblique lines through B_0, B_1, B_2, . . . are proportional to the ordinates y_0, y_1, y_2, . . . , so that these oblique lines are tangent lines to the required integral curve at these points. We can thus get a more accurate construction of the integral curve by drawing the curve through the points B_0, B_1, B_2, . . . , tangent to the oblique lines through these points.

The *polar distance* $SO = a$ is constructed with the same scale unit as the abscissa x, and the ordinate y' is measured with the same scale unit as the ordinate y.

Example. Determination of the mean spherical candle-power of a mazda lamp. — In testing a lamp for the *m. s. c. p.*, the intensity of illumination is measured every 15° by means of a rotating lamp and a photometer. The following table gives such measurements for a particular case:

Angle $\theta°$	0	15	30	45	60	75	90	105	120	135	150	165	180
c-p	11.55	13.0	15.4	22.4	31.0	38.8	42.7	43.9	45.2	32.0	21.8	9.1	0

According to the well-known Rousseau diagram, a semicircle is drawn (Fig. 105*h*) and divided into 15° sections, and perpendiculars are dropped from the points of division to the diameter, x_0, x_1, x_2, . . . , x_{12}. Upon these perpendiculars the values of *c-p* are laid off as ordinates. The area under the curve $A_0A_1A_2$ . . . A_{12} determined by these ordinates divided by the length of the base is the *m. s. c. p.* of the lamp, and this value multiplied by 4π will give the flux in lumens.

To measure the required area we have constructed the integral curve (Fig. 105*h*) by the method described above. We chose 7 in. for the length of the diameter of the circle and 1 in. = 10 *c-p* in laying off the ordinates. The y-axis or axis to which the horizontals are extended is drawn 5 in. to the right of the point A_0, so that the polar distance is $A_0O = a = 5$ in.

The area under the curve $A_0A_1A_2$ . . . A_{12} is measured by the ordinate $x_{12}B_{12} = 4.66$. Since $y' = \frac{1}{a} \times$ area, therefore area $= a \times y' = 5 \times 4.66 = 23.3$ sq. in. Since 1 in. on the scale of ordinates represents 10 *c-p* and the base of the diagram is 7 in., the *m. s. c. p.* $= \frac{23.3 \times 10}{7} = 33.3$ *c-p*. The straight line A_0B_{12} will cut the y-axis in a point D such that OD read on the *c-p* scale will also give the *m. s. c. p.*, for

$$\frac{OD}{A_0O} = \frac{x_{12}B_{12}}{A_0x_{12}}, \quad \text{or} \quad OD = \frac{x_{12}B_{12} \times a}{\text{base}} = \frac{\text{area}}{\text{base}} = m.\ s.\ c.\ p.$$

We measure $OD = 33.0$ *c-p.*

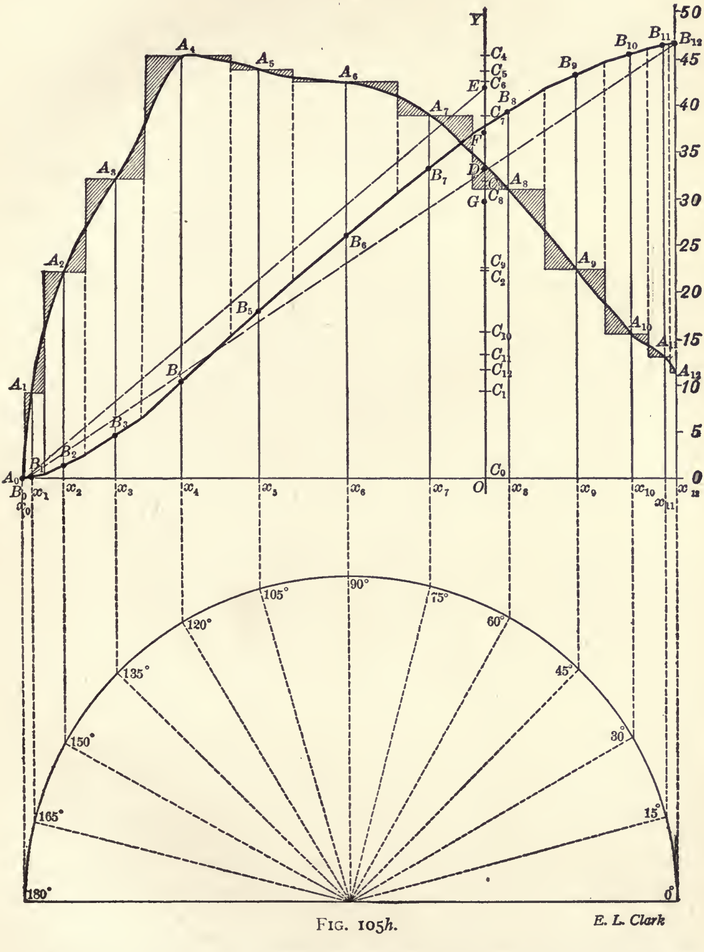


FIG. 105*h*. *E. L. Clark*

Having drawn the integral curve we may immediately find the *m. s. c. p.* of any portion of the lamp between two sections. Thus, for 15°

on each side of the vertical, the *m. s. c. p.* is found by drawing A_0E parallel to B_5B_7 and reading $OE = 42.0$ *c-p* on the candle-power scale, since

$$\frac{OE}{a} = \frac{x_7B_7 - x_5B_5}{x_5x_7}, \quad \text{or} \quad OE = \frac{\text{area under } A_5A_7}{\text{base}} = m.\,s.\,c.\,p.$$

Similarly the *m. s. c. p.* of the section above a horizontal plane through the lamp is measured by $OF = 37.0$ *c-p*, and the *m. s. c. p.* of the section below a horizontal plane through the lamp is measured by $OG = 29.5$ *c-p*.

106. Graphical differentiation. — If the integral curve $y' = f(x)$ is given we may construct the derivative curve $y = \dfrac{dy'}{dx}$ by using the principle that the ordinate of the derivative curve at any point P (x, y) (Fig. 106*a*) is equal to the slope of the integral curve or of the tangent line $P'T$ at the corresponding point P' (x, y').

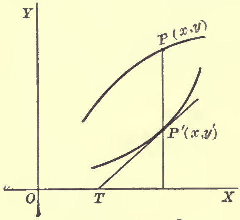


FIG. 106*a*.

The practical construction of the derivative curve consists of the following steps: (1) Divide the interval from x_0 to x_n (Fig. 106*b*) into n parts and erect the ordinates y_0', y_1', y_2', . . . , y_n'. (2) Construct the tangents at B_0, B_1, B_2, . . . , B_n and measure their slopes. (3) At x_0, x_1, . . . , x_n erect ordinates $x_0A_0 = y_0$, $x_1A_1 = y_1$, . . . , $x_nA_n = y_n$, where

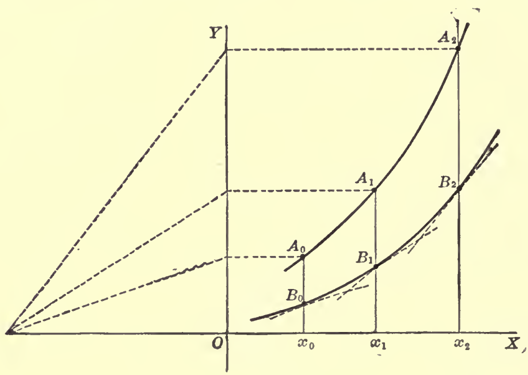


FIG. 106*b*.

the y's are proportional to the corresponding slopes, and draw a smooth curve through the points A_0, A_1, A_2, . . . , A_n. This curve will approximate the required derivative curve.

Example. The following table gives the pressure p in pounds per sq. in. of saturated steam at temperature $\theta°$ F. Construct the curve showing the rate of change of pressure with respect to the temperature, $dp/d\theta$.

θ	p	Δp	$\Delta\theta$	$\Delta p/\Delta\theta$
302.7	70			
307.4	75	5	4.7	1.06
311.8	80	5	4.4	1.14
316.0	85	5	4.2	1.19
320.0	90	5	4.0	1.25
323.9	95	5	3.9	1.28
327.6	100	5	3.7	1.35
331.1	105	5	3.5	1.43
334.5	110	5	3.4	1.47
337.8	115	5	3.3	1.52

In the above table we have approximated $dp/d\theta$ by $\Delta p/\Delta\theta$, *i.e.*, we have replaced the (θ, p) curve by a series of chords, and the slopes of the tangents by the slopes of these chords. We then plotted $(\theta, \Delta p/\Delta\theta)$ and joined the points by a smooth curve (Fig. 106*c*).

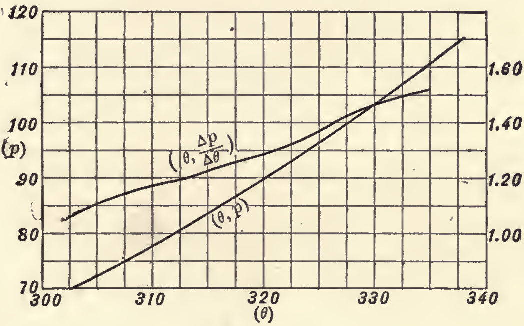


FIG. 106*c*.

It is evident that the difficulty in the construction of the derivative curve lies in the construction of the tangent line to the integral curve. The direction of the tangent line at any point is not very well defined by the curve. As a rule it is better to draw a tangent of a given direction and then mark its point of contact than to mark a point of contact and then try to draw the tangent at this point. A strip of celluloid on the under side of which are 2 black dots about 2 m.m. apart may be moved over the paper so that the two dots coincide with points on the integral curve and so that the secant line which they determine is practically identical with the tangent line. If the arc AB (Fig. 106*d*) is approxi-

mately the arc of a parabola, we have a more accurate construction of the tangent; the line joining the middle points M and M' of two parallel chords AB and $A'B'$ intersects the curve in P, the point of contact, and the tangent PT is parallel to the chord AB.

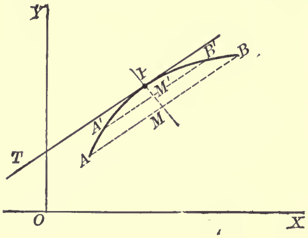


FIG. 106*d*.

We may also draw the derivative curve by purely graphical methods. The process is the reverse of the process described for constructing the integral curve (Art. 105). Let B_0, B_1, B_2, . . . be the points of contact of tangent lines to the integral curve (Fig. 105*g*). Choose a fixed point S at a convenient distance a to the left of the y-axis and draw the lines SC_0, SC_1, SC_2, . . . , parallel respectively to the tangent lines at B_0, B_1, B_2, ... Project the points C_0, C_1, C_2, . . . , horizontally on the ordinates at B_0, B_1, B_2, . . . , cutting these ordinates in A_0, A_1, A_2, ... The points A_0, A_1, A_2, . . . , are then points on the required derivative curve, since $B_0A_0 \div a =$ slope of $SC_0 =$ slope of tangent at B_0, etc. We may now join the points A_0, A_1, A_2, . . . by a smooth curve, or we may get greater accuracy by using the stepped line of horizontals and verticals. Thus, we draw the horizontals through the points A_0, A_1, A_2, . . . , and the verticals through the points of intersection of consecutive tangents to the integral curve. The arcs A_0A_1, A_1A_2, . . . , are now drawn so that the areas bounded by each arc, the horizontals, and the vertical, are equal.

107. Mechanical integration.* The planimeter. — This is an instrument for measuring areas. Consider a line PQ of fixed length l moving in any manner whatever in the plane of the paper. The motion of the line at any instant may be thought of as a motion of translation combined with a motion of rotation. Suppose the line PQ sweeps out the elementary area $PQQ'P' = dS$ (Fig. 107*a*). This may be broken up into a motion of translation of PQ to $P''Q'$ and a motion of rotation from $P''Q'$ to $P'Q'$. If dn is the perpendicular distance between the parallel positions PQ and $P'Q'$ and $d\phi$ is the angle between $P''Q'$ and $P'Q'$, then

$$dS = l\,dn + \tfrac{1}{2}\,l^2\,d\phi.$$

* For descriptions and discussions of various mechanical integrators see: Abdank-Abakanowicz, *Les Intégraphes* (Paris, Gauthier-Villars); Henrici, *Report on Planimeters* (Brit. Assoc. Ann. Rep., 1894, p. 496); Shaw, *Mechanical Integrators* (Proc. Inst. Civ. Engs., 1885, p. 75); *Instruments and Methods of Calculation* (London, G. Bell & Sons); Dyck's *Catalogue*; Morin's *Les Appareils d'Intégration*.

Now if PQ carries a rolling wheel W, called the integrating wheel, whose axis is parallel to PQ (Fig. 107*b*), then, while PQ moves to the parallel position $P''Q'$, any point on the circumference of this wheel receives a displacement dn, and while $P''Q'$ rotates to the position $P'Q'$, this point receives a displacement $a\,d\phi$, where a is the distance from Q to the plane

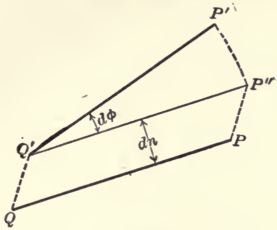


FIG. 107*a*.

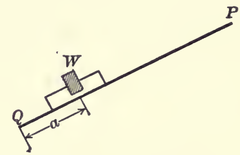


FIG. 107*b*.

of the wheel. So that, as PQ sweeps out the elementary area dS, any point on the circumference of the wheel receives a displacement

$$ds = dn + a\,d\phi.$$

Therefore,

$$dS = l\,ds - al\,d\phi + \tfrac{1}{2}\,l^2\,d\phi.$$

Hence the total area swept out by PQ is

$$S = l\int ds - al\int d\phi + \tfrac{1}{2}\,l^2\int d\phi.$$

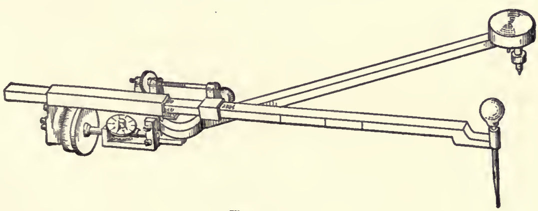


FIG. 107*c*.

Now, if PQ comes back to its original position without turning completely around, then the total angle of rotation $\int d\phi = 0$, so that

$$S = ls,$$

where s is the total displacement of any point on the circumference of the integrating wheel.

But if PQ comes back to its original position after turning completely around, then

$$S = ls - 2\,\pi al + \pi l^2.$$

The most common type of planimeter is the *Amsler polar planimeter* * (Fig. 107*c*). Here, Fig. 107*d*, by means of a guiding arm OQ, called the polar arm, one end Q of the tracer arm PQ is constrained to move in a circle while the other end P is guided around a closed curve *c-c-c-* . . . which bounds the area to be measured. Then the area $Q'P'PP''Q''QQ'$ is swept out twice but in opposite directions and the corresponding displacements of the integrating wheel cancel, so that the final displacement gives only the required area *c-c-c-* . . . . The circumference of the wheel is graduated so that one revolution corresponds to a certain definite number of square units of area.

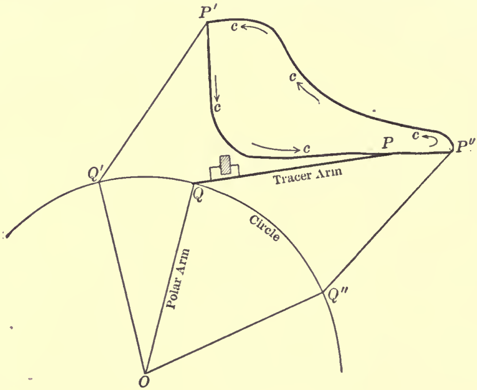


FIG. 107*d*.

The ordinary planimeter used for measuring indicator diagrams has $l = 4$ in. and the circumference of the wheel is 2.5 in.; hence one revolution corresponds to $4 \times 2.5 = 10$ sq. in. The wheel is graduated into 10 parts, each of these parts again into 10 parts, and a vernier scale allows us to divide each of the smaller divisions into 10 parts, so that the area can be read to the nearest hundredth of a sq. in. The indicator diagram on p. 228 gives a planimeter reading of 2.55 sq. in., which agrees with the result found by Simpson's rule with 15 ordinates.

The polar planimeters used in the work in Naval Architecture usually have a tracer arm of length 8 in., and a wheel of circumference 2.5 in., so that one revolution corresponds to 20 sq. in., thus giving a larger range for the tracing point. If the area to be measured is quite large, it may be split up into parts and the area of each part measured; or the area may be re-drawn on a smaller scale and the reading of the wheel multiplied by the area-scale of the drawing.†

* This instrument was first put on the market by Amsler in 1854.

† If PQ (Fig. 107 *d*) turns completely around, the required area is $S + \pi\ (OQ)^2$.

If very accurate results are required, account must be taken of several errors. (1) The axis of the integrating wheel may not be parallel to the tracer arm PQ. This error can be partly eliminated by taking the mean of two readings, one with the pole O to the left of the tracer arm, the other with the pole to the right * (Fig. 107*e*). This cannot be done with the ordinary Amsler planimeter because the tracer arm is mounted above the polar arm, but can be done with any of the Coradi or Ott *compensation planimeters;* one of these instruments is illustrated in Fig. 107*f*. (2) The integrating wheel may slip; some of this slipping may be due to the irregularities of the paper and has been obviated by the use of *disc planimeters*, in which the recording wheel works on a revolving disc instead of on the surface of the paper.

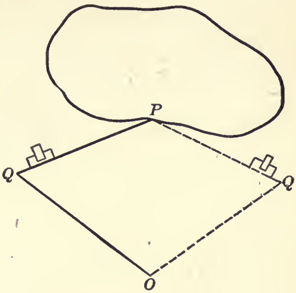


FIG. 107*e*.

Various types of *linear planimeters* have been constructed. These differ from the polar planimeters in that one end of the tracer arm is constrained to move in a straight line instead of in a circle. Planimeters of the linear type form part of the integrators described in Art. 108.

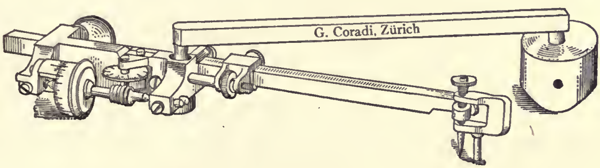


FIG. 107*f*.

Various other types of planimeters have been constructed, which do not have an integrating wheel. One of the best known of these is that of Prytz, also known as the hatchet planimeter.† In this form of the instrument (Fig. 107*g*) the end Q forms a knife-edge so that Q can only move freely along the line PQ. When P traces the given curve, Q will describe a curve such that PQ is always tangent to it.

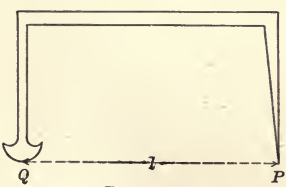


FIG. 107*g*.

* For a proof of this statement, see Instruments and Methods of Calculation, p. 196.

† For the theory of this instrument, see F. W. Hill, Phil. Mag., xxxviii, 1894, p. 265.

Prytz starts the instrument with the point P approximately at the center of gravity G of the area to be measured, moves P along the radius vector to the curve, completely around the curve, and back along the same radius vector to G. The required area is then given approximately by $l^2\phi$, where l is the length PQ and ϕ is the angle between the initial and final positions of the line PQ.

108. Integrators. — The *Amsler integrator* is practically an extension of the linear planimeter. In the latter instrument, the end Q of the tracer arm PQ of constant length l, is constrained to move in a straight line $X'X$, while the tracing point P describes a circuit of the curve. If the axis of the integrating wheel attached to PQ makes a variable angle $m\alpha$ with $X'X$ (Fig. 108*a*) at each instant, the point P will have for ordinate $y_m = l \sin m\alpha$, and the area described by P will be $\int l \sin m\alpha \, dx$. On the other hand, the area described by P is equal to l times the displacement of any point on the circumference of the integrating wheel; hence $\int \sin m\alpha \, dx$ *is equal to the displacement of a point on the circumference of an integrating wheel whose axis makes an angle* $m\alpha$ *with* $X'X$.

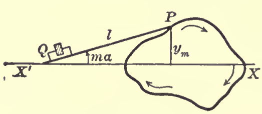


FIG. 108*a*.

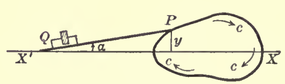


FIG. 108*b*.

Now, given a curve c-c-c- . . . (Fig. 108*b*),

$$\text{Area} = \int y\,dx = \int l \sin\alpha\,dx = l\int \sin\alpha\,dx.$$

$$\begin{aligned}\text{Moment of area about } X'X &= \frac{1}{2}\int y^2\,dx = \frac{1}{2}\int l^2 \sin^2\alpha\,dx = \frac{l^2}{4}\int (1-\cos 2\alpha)\,dx\\ &= \frac{l^2}{4}\int dx - \frac{l^2}{4}\int \sin(90^\circ - 2\alpha)\,dx\\ &= -\frac{l^2}{4}\int \sin(90^\circ - 2\alpha)\,dx, \quad \text{since} \quad \int dx = 0,\end{aligned}$$

the arm PQ returning to its original position when P makes a complete circuit of the curve.

$$\begin{aligned}\text{Moment of inertia of area about } X'X &= \frac{1}{3}\int y^3\,dx = \frac{1}{3}\int l^3 \sin^3\alpha\,dx = \frac{l^3}{3}\int\left(\frac{3}{4}\sin\alpha - \frac{1}{4}\sin 3\alpha\right)dx\\ &= \frac{l^3}{4}\int \sin\alpha\,dx - \frac{l^3}{12}\int \sin 3\alpha\,dx.\end{aligned}$$

Now, $\int \sin \alpha \, dx$, $\int \sin (90° - 2\alpha) \, dx$, and $\int \sin 3\alpha \, dx$, and hence the area, moment, and moment of inertia can be measured by three integrating wheels whose axes at any instant make angles α, $90° - 2\alpha$, and 3α, respectively, with $X'X$.

The *Amsler 3-wheel integrator* (Fig. 108*c*) consists of an arm PQ and 3 integrating wheels A, M, and I. The instrument is guided by a carriage which rolls in a straight groove in a steel bar; this bar may be set at a proper distance from the hinge of the tracer arm by the aid of trams. The

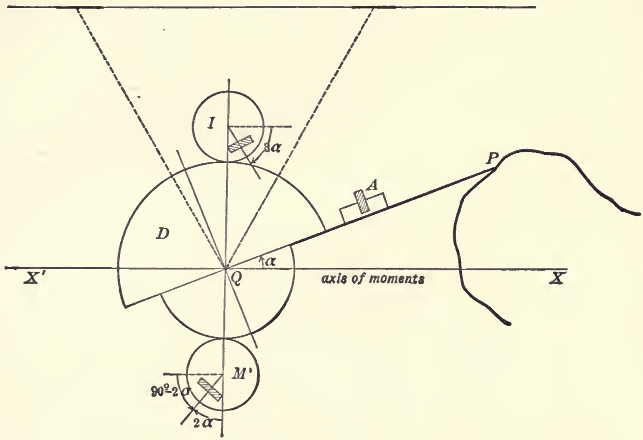


FIG. 108*c*.

line $X'X$, which passes through the points of the trams and under the hinge, is the axis about which the moment and moment of inertia are measured. The radius of the disk containing the M-wheel is one-half the radius, and the radius of the disk containing the I-wheel is one-third the radius of the circular disk D to which they are geared. Therefore, the axis of the M-wheel turns through twice, and the axis of the I-wheel turns through three times the angle through which the tracer arm PQ or the axis of the A-wheel swings from the axis $X'X$.

The integrating wheels are set so that in the initial position, *i.e.*, when P lies on $X'X$, the axes of the A- and I-wheels are parallel to $X'X$ while the axis of the M-wheel is perpendicular to $X'X$. Then, when the tracer arm PQ makes an angle α with $X'X$, the axes of the A-, M-, and I-wheels make angles α, $90° - 2\alpha$, and 3α, respectively, with $X'X$. Furthermore, the graduations of the M-wheel are marked so that these graduations move backward while the graduations on the other wheels move

forward. Hence, when P has completed the circuit, and if a, m, and i are the displacements of points on the circumferences of the A-, M-, and I-wheels, respectively, we have

$$Area = la; \quad Moment = \frac{l^2}{4}m; \quad Moment\ of\ Inertia = \frac{l^3}{4}a - \frac{l^3}{12}i.$$

The wheels are graduated from 1 to 10 so that a reading of 5, for example, means 5/10 of a revolution. The constants by which these readings are multiplied depend upon the length of the tracing arm and the circumferences of the integrating wheels. In the ordinary instrument, $l = 8$ in. and the circumferences of the A-, M-, and I-wheels are

$$C_A = 2.5 \text{ in.}, \qquad C_M = 2.5 \text{ in.}, \qquad C_I = 2.34375 \text{ in.}$$

Thus, to find the

area,	a must be multiplied by	$8 \times 2.5 = 20;$
moment,	m " " "	" $\frac{8^2}{4} \times 2.5 = 40;$
moment of inertia,	a " " "	" $\frac{8^3}{4} \times 2.5 = 320,$
and	i " " "	" $\frac{8^3}{12} \times 2.34375 = 100.$

Finally, if a_1, a_2, m_1, m_2, and i_1, i_2 are the initial and final readings of the A-, M-, and I-wheels, we have

$$Area = 20\,(a_2 - a_1); \quad Moment = 40\,(m_2 - m_1);$$
$$Moment\ of\ Inertia = 320\,(a_2 - a_1) - 100\,(i_2 - i_1).$$

109. The integraph. — This is a machine which draws the integral curve, $y' = \int f(x)\,dx$, of the curve $y = f(x)$. The most familiar type of such machines is the one invented by Abdank-Abakanowicz in 1878. The theory of its construction is very simple. A diagram of the machine is given in Fig. 109*a*. The machine is set to travel along the base line of the curve to be integrated, and two non-slipping wheels, W, ensure that the motion continues along this axis. The scale-bar slides along the main frame as the tracing point P, at the end of the bar, describes the curve $y = f(x)$ to be integrated. The radial-bar turns about the point Q which is at a constant distance a from the main frame. The motion of the recording pen at P_1 is always parallel to the plane of a small, sharp-edged, non-slipping wheel w, and by means of the parallel frame-work $ABCD$, the plane of the wheel w is maintained parallel to the radial bar [since w is set perpendicular to AB which is parallel and equal to CD throughout the motion, and the radial bar is set perpendicular to CD]. As the point P describes the curve $y = f(x)$, the angle θ between the radial-bar and the

axis, and consequently the angle θ between the plane of the wheel and the axis, are constantly changing, and the recording pen at P_1 draws a curve with ordinate y' such that its slope

$$\frac{dy'}{dx} = \tan\theta = \frac{y}{a} = \frac{f(x)}{a},$$

and therefore,

$$y' = \frac{1}{a}\int f(x)\,dx = \frac{1}{a} \times \text{area } ORP,$$

so that the curve drawn by P_1 is the integral curve of the curve traced by P.

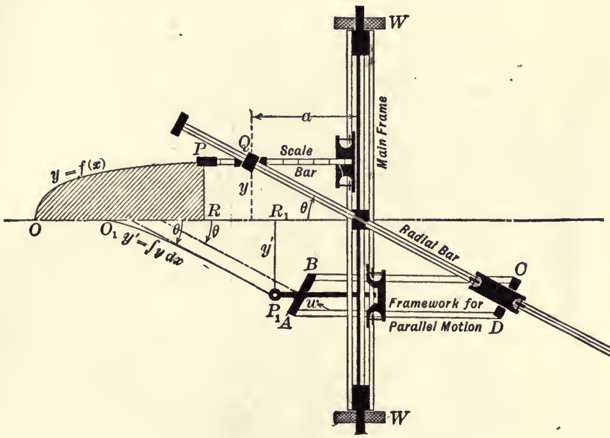


FIG. 109*a*.

If we now set the machine so that the point P traces the integral curve, then the recording pen P_1 will draw its integral curve

$$y'' = \int y'\,dx = \int\left(\int y\,dx\right)dx = \int\int y\,dx^2.$$

We may thus draw the successive integral curves y', y'', y''', ... Fig. 109*b* gives the integral curves connected with the curve of loads of the shaft of a Westinghouse-Rateau Turbine. The curve of loads is represented by the broken line in the figure. By successive integration we get the shear curve, the bending moment curve, the slope curve, and the deflection curve. The distance marked "offset" is the distance OO_1 in Fig. 109*a*.

SHAFT OF THE WESTINGHOUSE-RATEAU TURBINE

Scale for Shaft Length, 5/16 In.=10 Ins.
" " Loads, 5/16 In.=500 lbs.

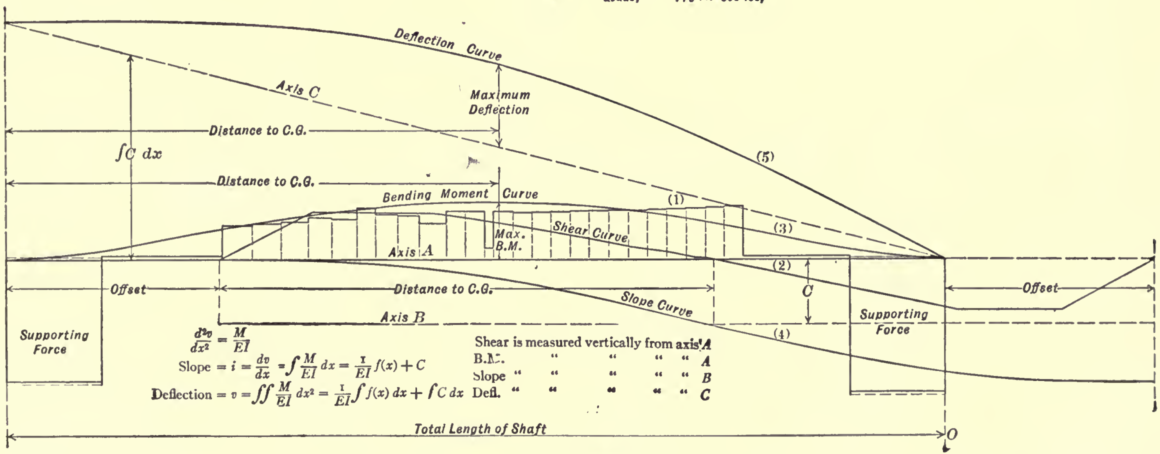


FIG. 109b.

110. Mechanical differentiation. The Differentiator. — This is a machine which draws the derivative curve $y' = \frac{dy}{dx}$ of the curve $y = f(x)$. Since the ordinate of the derivative curve is equal to the slope of the integral curve, it is necessary to construct the tangent lines at a series of points of the integral curve. We have already mentioned (Art. 106) the use of a strip of celluloid with two black dots on its under side to deter-

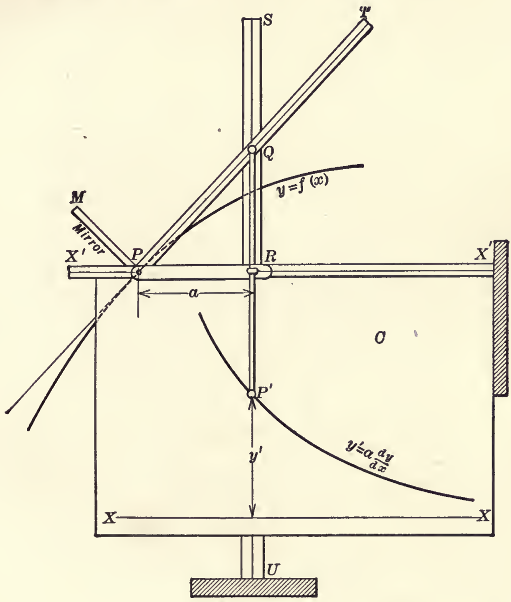


FIG. 110.

mine the direction of the tangent. This scheme is used in a differentiating machine constructed by J. E. Murray.* In a differentiating machine recently constructed by A. Elmendorf,† a silver mirror is used for determining the tangent. The mirror is placed across the curve so that the curve and its image form a continuous unbroken line, for then the surface of the mirror will be exactly normal to the curve, and a perpendicular to this at the point of intersection of the mirror and the curve will give the direction of the tangent line. If the surface of the mirror de-

* Proc. Roy. Soc. of Edinburgh, May, 1904.

† Scientific American Supplement, Feb. 12, 1916.

viates even slightly from the normal, a break will occur at the point where the image and curve join. It is claimed that with a little practice a remarkable degree of accuracy can be obtained in setting the mirror.

Fig. 110 gives a diagram illustrating the working of this machine. The tracing point P follows the curve $y = f(x)$ so that the curve and its image in the mirror MP form a continuous unbroken line; then the arm PT, which is set perpendicular to the mirror, will take the direction of the tangent line to the curve. The link PR, of fixed length a, is free to move horizontally in the slot $X'X'$ of the carriage C. The vertical bar SU passes through R and is constrained to move horizontally by heavy rollers. The point Q slides out along the tangent bar PT and also vertically in the bar SU, carrying with it the bar QP'. If we choose for the x-axis a line XX whose distance from $X'X'$ is equal to QP', then the point P' will draw a curve whose ordinate is equal to $y' = RQ$. But RQ/a is the slope of the tangent PT, hence, $y' = a \times \frac{dy}{dx}$, and the curve drawn by P' is the derivative curve of the curve traced by P.

The machine is especially useful for differentiating deflection-time curves to obtain velocity-time curves, and by a second differentiation, acceleration-time curves. It is also helpful in solving many other problems.

EXERCISES.

Apply the approximate rules of integration (Art. 101) to the following examples:

1. Evaluate $\int_{0.2}^{1.0} \frac{dx}{x}$, when $h = 0.1$, and when $h = 0.05$, and compare the results with the values obtained by integration.

2. Evaluate $\int_0^{\pi} \sin x\, dx$, when $h = \frac{\pi}{6}$, and when $h = \frac{\pi}{12}$, and compare the results with the values obtained by integration.

3. The arc of a quadrant of an ellipse whose eccentricity is 0.5 is given by $\int_0^{\frac{\pi}{2}} \sqrt{1 - 0.25 \sin^2 \phi}\, d\phi$. Evaluate the integral when $h = 9°$.

4. Evaluate $\int_0^3 \frac{dx}{\sqrt[3]{x^3 - x + 1}}$, when $h = 0.5$.

5. The semi-ordinates in ft. of the deck plan of a ship are

3, 16.6, 25.5, 28.6, 29.8, 30, 29.8, 29.5, 28.5, 24.2, 6.8;

these measurements are 28 ft. apart. Find the area of the deck.

6. Given the following data for superheated steam

v	2	4	6	8	10
p	105	42.7	25.3	16.7	13

Find the work done.

7. The length of an indicator diagram is 3.6 in. The widths of the diagram, 0.3 in. apart, are

0, 0.40, 0.52, 0.63, 0.72, 0.93, 0.99, 1.00, 1.00, 1.00, 1.00, 0.97, 0.

Find the mean effective pressure.

8. The length of an indicator diagram is 3.2 in. The widths of the diagram, 0.2 in. apart, are

1.00, 1.68, 1.62, 1.00, 0.64, 0.48, 0.36, 0.26, 0.

Find the mean effective pressure.

9. The speed of a car is v miles per hour at a time t seconds from rest;

t	0	5	10	15	20	25	30
v	0	3.7	7.5	10.9	13.0	13.7	14

Find the distance traversed in 30 seconds.

10. s is the specific heat of a body at temperature $\theta°$ C.

θ	0	2	4	6	8	10	12
s	1.00664	1.00543	1.00435	1.00331	1.00233	1.00149	1.00078

Find the total heat required to raise the temperature of a gram of water from 0° C. to 12° C. (total heat $= \int_{\theta_1}^{\theta_2} s\, d\theta$).

11. The areas in sq. ft. of the sections of a ship above the keel and two feet apart are

2690, 3635, 4320, 4900, 5400.

Find the total displacement in tons.

12. A reservoir is in the form of a volume of revolution and D is the diameter in ft. at a depth of p feet beneath the surface of the water.

p	0	16	32	48	64	80	96
D	110	105	100	86	66	48	27

Find the number of gallons of water the reservoir holds when full.

13. A plane board is immersed vertically in water. The widths of the board in ft. parallel to the surface of the water and at depths $\frac{1}{2}$ ft. apart are

4.0, 3.6, 3.0, 1.7, 1.3, 1.0, 0.8, 0.6, 0.1.

Find the pressure on the board and the depth of the center of pressure when the surface of the water is level with the top of the board.

14. The half-ordinates in ft. of the mid-ship section of a vessel at intervals 2 ft. apart are

12.2, 12.5, 12.6, 12.7, 12.7, 12.5, 12.1, 11.5, 10.1, 6.5, 0.2.

Find the position of the center of gravity of the section.

15. The shape of a quarter-section of a hollow pillar is given by the following table. The axes of x and y are the shortest and longest diameters.

x in.	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50
outside y_1 in.	6	5.95	5.90	5.83	5.76	5.64	5.48	5.22	4.99	4.68	4.35	3.88	3.25	2.34	0
inside y_2 in.	5	4.90	4.78	4.65	4.45	4.22	3.80	3.40	2.77	2.08	0				

Find the moments of inertia of the section about the x- and y- axes.

16. Apply the formulas for numerical differentiation (p. 235) to table (2) $y = x^3$ on p. 211, and find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 5.31$ and $x = 5.33$. Check the results by actual differentiation.

17. Apply the formulas for numerical differentiation (p. 235) to table (8) $y = \log \sin x$ on p. 212, and find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 1° 20'$ and $x = 1° 24'$. Check the results by actual differentiation.

18. In the following table, s is the distance in ft. which the projectile of a gun travels along the bore in t sec.

s	0	1	2	3	4	5	6	7	8	9	10
t	0	0.0360	0.0490	0.0598	0.0695	0.0785	0.0871	0.0953	0.1032	0.1109	0.1184

Find the velocity $v = \frac{ds}{dt} = 1 \Big/ \frac{dt}{ds}$, and the acceleration $a = \frac{d^2s}{dt^2} = -\frac{d^2t}{ds^2} \Big/ \left(\frac{dt}{ds}\right)^3$ when $s =$ 5 ft.

19. Use the data given in Ex. 6 to find the rate of change of the pressure with respect to the volume, dp/dv, when $v = 4$ and $v = 5$.

20. Use the data given in Ex. 9 to find the acceleration, $a = \frac{dv}{dt}$, when $t = 10$ and $t = 12$.

21. Find the minimum value of the polynomial which has the values

x	0	2	4	6
y	3	3	11	27

22. The following table gives the results of measurements made on a normal induction curve for transformer steel; B is the number of kilolines per sq. cm.; μ is the permeability.

B	1	2	3	4	5	6	7	8	9	10	11	12
μ	625	870	1035	1210	1350	1465	1520	1480	1430	1370	1280	1130

Find the maximum permeability.

23. Construct the integral curve of the parabola $y = x - \frac{1}{2} x^2$ as x varies from 0 to 2.

24. Construct the integral curve of the sine wave $y = 2 \sin 2x$ as x varies from 0 to π.

25. The following table gives the accelerations a of a body sliding down an inclined plane for various distances s in ft.

s	0	100	200	300	400	500	600	700	800	900	1000
a	0.320	0.304	0.256	0.176	0.080	−0.016	−0.080	−0.136	−0.176	−0.208	−0.240

Use the method employed in the illustrative example on p. 239 for drawing the integral curves and determining the velocity, $v = \sqrt{2 \int a\, ds}$, and the time, $t = \int \frac{1}{v}\, ds$, for any distance, if $v = 0$ and $t = 0$ when $s = 0$.

26. The following table gives the accelerations a of a body at various velocities v in ft. per sec.

v	0	1	2	3	4	5
a	0.405	0.360	0.283	0.179	0.069	0.013

Draw the integral curves to determine the time, $t = \int \frac{1}{a}\, dv$, and the distance, $s = \int \frac{v}{a}\, dv$, for any velocity, if $t = 0$ and $s = 0$ when $v = 0$.

27. In the following table

s	0	1	4	6	8	11.5	15	19	20
P	38,000	38,500	38,500	35,500	27,500	19,000	15,700	11,000	3850

P is the resultant pressure in pounds on the piston of a steam engine at distances s inches from the beginning of the stroke. Draw the integral curve to find the work done as the piston moves forward. $\left(\text{Work} = \int P\,ds.\right)$

28. A car weighs 10 tons. It is drawn by a pull of P lbs.; t is the time in seconds since starting.

t	0	2	5	8	10	13	16	19	22
P	1020	980	882	720	702	650	713	722	805

If the retarding friction is constant and equal to 410 lbs., draw the integral curve to find the speed of the car at any time. $\left(\text{Momentum} = \int (P - 410)\,dt.\right)$

29. In the following table

t	0.00490	0.00598	0.00695	0.00785	0.00871	0.00953	0.01032	0.01109	0.01184
v	869	987	1074	1142	1195	1242	1277	1309	1335

v is the velocity of projection in ft. per sec. in the bore of a gun at time t sec. from the beginning of the explosion. If $s = 2$ ft. when $t = 0.00490$ sec., draw the integral curve to show the relation between the distance and the time.

30. A beam 10 ft. long is loaded as in the following table, where w is the weight per unit length at distances x ft. from the free end.

x	0	1	2	3	4	5	6	7	8	9	10
w	2	2.5	3.7	5.5	7.7	9.7	11.2	12.2	11.8	10.2	7.2

Draw integral curves to show (1) the shearing force, $s = \int w\,dx$ and (2) the bending moment, $M = \int s\,dx$.

31. The following table gives the measurements for every 15° of the intensity of illumination of a lamp.

Angle $\theta°$	0	15	30	45	60	75	90	105	120	135	150	165	180
c-p	60.5	88.0	99.5	86.5	50.0	31.0	29.0	29.0	28.0	20.0	15.0	13.0	12.5

Apply the method of the illustrative example on p. 242 to find the *m.s.c.p.* for various sections of the lamp.

32. In the following table

t	0	10	20	30	40	50	60	70	80	90	100
s	0	156	608	1308	2180	3132	4076	4942	5676	6236	6588

s is the distance in ft. traversed by a body weighing 2000 lbs. in t sec. Draw the derivative curves to show the velocity and acceleration at any time. Also draw the curve showing the relation between the kinetic energy and the force.

33. The observed temperature θ in degrees Centigrade of a vessel of cooling water at time t in min. from the beginning of observation are given in the following table:

t	0	1	2	3	5	7	10	15	20
θ	92.0	85.3	79.5	74.5	67.0	60.5	53.5	45.0	39.5

Draw the derivative curve to show the rate of cooling at any time.

INDEX.